

A numerical investigation on the unilateral contact problem with friction

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SUMMARY: In this paper we develop a finite element analysis of the contact problem with friction of some one-dimensional models, like beams and axisymmetric plates, unilaterally constrained against on an elastic half-space, taking into account the shear deformation of these structures.

The investigation places emphasis on the influence of the more significative parameters, involved in the theory.

1. Introduction

During the last few years the frictionless unilateral contact problems have been an active subject of research and many interesting results have been obtained from a theoretical and numerical point of view [GLADWELL and IYER, 1974; SVEC, 1974; TOSCANO and MACERI, 1980; ASCIONE and GRIMALDI, 1984].

On the contrary, the engineering problems of unilateral contact with friction between a beam or a plate and an elastic foundation have been less investigated [PANAGIOTOPOULOS, 1975].

Previous numerical results relative to the effects of the friction in engineering problems refer mainly to the case of adhesive contact [SELVADURAI, 1979; BILOTTA, 1982].

From a theoretical point of view the equilibrium problem of an elastic body, unilaterally constrained with friction against a rigid obstacle is still an open problem, if a Coulomb law of friction is considered (Signorini's problem with friction) [DEVAUT and LIONS, 1972]. In fact, the proof of existence of solutions of this problem is known only for some very special cases [NECAS *et al.*, 1980]. The source of difficulties, as pointed out by DUVAUT [1980] is the lack of smoothness of the normal contact pressure.

To overcome this obstacle some special friction laws were introduced by ODEN and PIRES [1981; 1982].

Recently, the unilateral contact problem with friction between a plate and an elastic half-space was analyzed by ASCIONE and BRUNO [1985a; b], taking into account a friction law like a regularization of the classical Coulomb one. This law makes possible to furnish a mathematical formulation of the problem in which

results of existence, uniqueness, and approximation of the solution are early given.

The aim of this present paper is a numerical investigation via finite elements about some contact problems with friction, according to the model suggested by Ascione and Bruno.

In particular some one-dimensional examples of beam and axisymmetric circular plates, taking into account shear deformations of these structures, are examined.

The investigation places emphasis on the influence of the more significative parameters, with particular interest to the choice of the interpolating functions.

Moreover the dependence on the results of the parameter of regularization ε , simulating a Coulomb behaviour, for $\varepsilon \rightarrow 0$, is examined.

2. Formulation of the problem

In this section we consider the statical equilibrium problem of a Mindlin plate unilaterally constrained with friction on an elastic, homogeneous and isotropic half-space (Fig. 1).

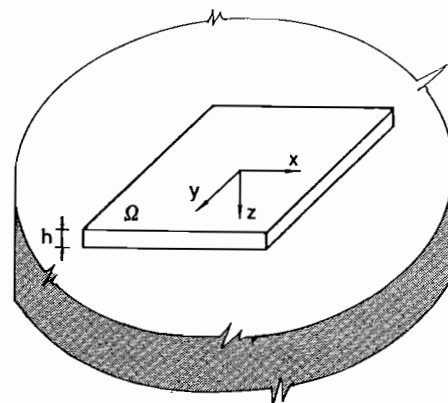


Fig. 1

According to Mindlin's theory the plate displacement components can be put in the form:

$$u_x(x, y, z) = -z\psi_x(x, y) + u_s(x, y) \quad (2.1a)$$

$$u_y(x, y, z) = -z\psi_y(x, y) + v_s(x, y) \quad (2.1b)$$

$$u_z(x, y, z) = w(x, y) \quad (2.1c)$$

where ψ_x, ψ_y are the bending slopes along the x and y axis and u_s, v_s are the extensional displacements of the middle plane.

Therefore, it is assumed that the plane sections remains plane after the deformation but not necessarily normal to the midplane.

We assume the friction law proposed in ASCIONE and BRUNO [1985a] which is depicted for an one-dimensional example in Fig. 2.

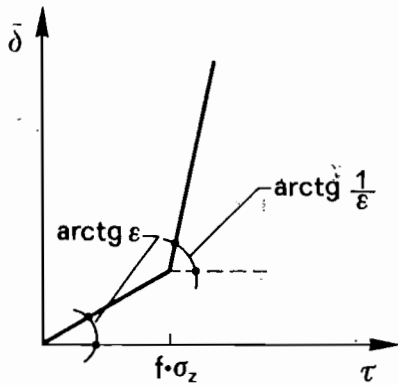


Fig. 2

From Fig. 2 we observe that the relative displacement δ , between structure and foundation is always different from zero even if its magnitude becomes considerable just when the shear stress exceeds the limit value $f\sigma_z$. This does not agree with Coulomb's but law it is justifiable from the physical point of view

since the relative displacement shows itself really for low values of the shear stress.

Let the following now be:

- $V = H^1(\Omega)$: Sobolev space of order 1 on Ω ,
- $V = V^3$: space of admissible configuration $\underline{v} = (u_s, v_s, w, \psi_x, \psi_y)$ of the plate,
- V' : dual space of V ,
- $\langle \dots \rangle$: duality pairing between dual spaces,
- $\underline{Q} = V^3$: space of the generalized loads corresponding to the forces \underline{q} acting on the plate upper surface,
- $\underline{\Sigma} = V^3$: space of the generalized (loads) \underline{g} acting on the plate lower surface,
- \underline{V}' : space of the generalized loads associated to the kinematic variable \underline{v} ,
- $\underline{R}^- : \underline{Q} \rightarrow \underline{V}', \underline{R}^+ : \underline{\Sigma} \rightarrow \underline{V}'$: linear, injective and continuous operators from V^3 into \underline{V}' defined as:

$$\langle \underline{R}^- \underline{q}, \underline{v} \rangle = \langle q_x, u_s \rangle + \langle q_y, v_s \rangle + \langle q_z, w \rangle + h/2 \langle q_x, \psi_x \rangle + h/2 \langle q_y, \psi_y \rangle, \quad (2.2a)$$

$$\langle \underline{R}^+ \underline{g}, \underline{v} \rangle = \langle \sigma_x, u_s \rangle + \langle \sigma_y, v_s \rangle + \langle \sigma_z, w \rangle - h/2 \langle \sigma_x, \psi_x \rangle - h/2 \langle \sigma_y, \psi_y \rangle, \quad (2.2b)$$

$\underline{A} : \underline{V} \rightarrow \underline{V}'$: equilibrium plate operator associated to the bilinear form:

$$\begin{aligned} \langle \underline{A}\underline{v}, \delta\underline{v} \rangle = & C \int_{\Omega} \left[\frac{\partial u_s}{\partial x} \frac{\partial \delta u_s}{\partial x} + \frac{\partial v_s}{\partial y} \frac{\partial \delta v_s}{\partial y} + \nu_p \left(\frac{\partial u_s}{\partial x} \frac{\partial \delta v_s}{\partial y} + \frac{\partial \delta u_s}{\partial x} \frac{\partial v_s}{\partial y} \right) + \right. \\ & \left. + \frac{1 + \nu_p}{2} \left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} \frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} \right) \right] d\Omega + \\ & + D \int_{\Omega} \left[\frac{\partial \psi_x}{\partial x} \frac{\partial \delta \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \frac{\partial \delta \psi_y}{\partial y} + \nu_p \left(\frac{\partial \psi_x}{\partial x} \frac{\partial \delta \psi_y}{\partial y} + \frac{\partial \delta \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} \right) + \right. \\ & \left. + \frac{1 + \nu_p}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \left(\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} \right) \right] d\Omega + \\ & + kG_p h \int_{\Omega} \left[\left(\frac{\partial w}{\partial x} - \psi_x \right) \left(\frac{\partial \delta w}{\partial x} - \delta \psi_x \right) + \left(\frac{\partial w}{\partial y} - \psi_y \right) \left(\frac{\partial \delta w}{\partial y} - \delta \psi_y \right) \right] d\Omega \quad \forall \underline{v}, \delta\underline{v} \in \underline{V}; \end{aligned} \quad (2.3)$$

$V_f = H^{1/2}(\Omega)$: Sobolev space of order 1/2 on Ω ,
 $\underline{V}_f = V_f^3$: space of the admissible boundary displacements of the elastic half-space,
 $\underline{T} : \underline{V}_f' \rightarrow \underline{V}_f$: linear operator which gives the restrictions of the elastic half-space boundary displacements v_f to Ω as a function of the interaction σ between plate and half-space,
 $\underline{K}_f = \{ \underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \in V_f' : \langle \sigma_z, w_f \rangle > 0, w_f \geq 0 \}$ convex set of the admissible reaction $\underline{\sigma}$ acting on the elastic half-space,
 $\| \cdot \|_X$: norm symbol on the space X .

The linear operator \underline{T} is continuous and coercive on \underline{V}_f' , while the linear operator \underline{A} is continuous and positive semidefinite on \underline{V} , i.e:

$$\| \underline{A} \underline{v} \|_{\underline{V}} \leq c \| \underline{v} \|_{\underline{V}} \quad (2.4a)$$

(continuity)

$$\| \underline{T} \underline{\sigma} \|_{\underline{V}_f} \leq c_f \| \underline{\sigma} \|_{\underline{V}_f'} \quad (2.4b)$$

$c, c_f > 0, \forall \underline{v} \in \underline{V} \text{ and } \forall \underline{\sigma} \in \underline{V}_f'$;

$$\langle \underline{\sigma}, \underline{T} \underline{\sigma} \rangle \geq c'_f \| \underline{\sigma} \|^2_{\underline{V}_f'} \quad (2.5a)$$

(coerciveness)

$$\langle \underline{A} \underline{v}, \underline{v} \rangle \geq 0, \quad (2.5b)$$

(positive semidefinition)

$$c'_f > 0, \forall \underline{\sigma} \in V_f' \text{ and } \forall \underline{v} \in V.$$

As a consequence of the assumed friction law, the relative displacement δ in the (x, y) -plane between plate and half-space is given as

$$\underline{\delta} = \mathcal{J}_0 \varepsilon \underline{J} \underline{\tau} + \frac{1-\varepsilon^2}{\varepsilon} (|\underline{J} \underline{\tau}| - f |\sigma_z|)^+ \frac{\underline{J} \underline{\tau}}{|\underline{J} \underline{\tau}|}, \quad (2.6)$$

where

J is the canonical isometry from V_f' onto V_f ,

\underline{J} is the canonical isometry from \underline{V}_f' onto \underline{V}_f ,

$$|\underline{u}| = (u_x^2 + u_y^2 + u_z^2)^{1/2} \quad \forall \underline{u} \in \underline{V}_f,$$

$(x)^+ = \max \{ 0, x \}$ for each real number x ,

ε, f are two assigned positive constants.

The other contact conditions are:

$$\delta_z = (w_f - w) \geq 0, \quad (2.7a)$$

$$\langle \sigma_z, \delta_z \rangle = 0. \quad (2.7b)$$

After these preliminaries, the unilateral contact problem with friction between plate and half-space can be formulated as follows:

« Find $(\underline{v}, \underline{\sigma}) \in \underline{V} \times \underline{K}'$ such that:

$$\langle \underline{A} \underline{v} - \underline{R}^- \underline{q} + \underline{R}^+ \underline{\sigma}, \delta \underline{v} \rangle = 0 \quad (2.8a)$$

$$\langle \delta \underline{\sigma} - \underline{\sigma}, \underline{R}' \underline{v} - \underline{T} \underline{\sigma} - \mathcal{J} \varepsilon(\underline{\sigma}) \rangle \leq 0, \quad (2.8b)$$

where \underline{R}' is the dual operator of \underline{R}^+ .

We observe that eq. (2.8a) is the virtual work equation, while eq. (2.8b) characterizes the contact conditions (2.6) and (2.7) between plate and half-space.

3. Variational formulation

The following auxiliary problem, characterized by a friction law independent of the unknown normal stress σ_z (Fig. 3), is very helpful to study the problem (2.8).

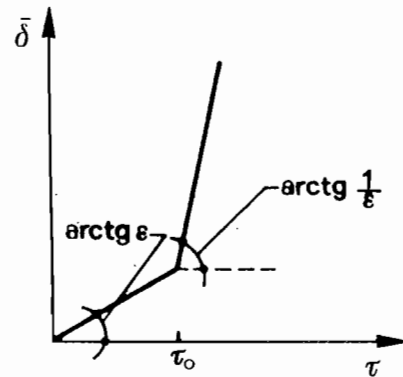


Fig. 3

In other words the relationship between the relative displacement δ and the shear tension τ becomes:

$$\underline{\delta} = \mathcal{J}_0 \varepsilon(\sigma) = \varepsilon \underline{J} \underline{\tau} + \frac{1-\varepsilon^2}{\varepsilon} (|\underline{J} \underline{\tau}| - \tau_0)^+ \frac{\underline{J} \underline{\tau}}{|\underline{J} \underline{\tau}|}, \quad (3.1)$$

where τ_0 is an assigned positive function of $V_f' (\tau_0 \geq 0)$.

It is easy to recognize that the operator $\mathcal{J}_0 \varepsilon$ is the weak differential of the quadratic functional:

$$\Phi_{0\varepsilon}(\underline{\sigma}) = \frac{1}{2} \varepsilon \| \underline{J} \underline{\tau} \|_0^2 + \frac{1-\varepsilon^2}{2\varepsilon} \| (|\underline{J} \underline{\tau}| - \tau_0)^+ \|_0^2, \quad (3.2)$$

where $\| \cdot \|_0$ is the L^2 -norm.

We observe that the equations of the auxil-

lary problem can be characterized as the stationarity points of the saddle functional

$$F_0(\underline{v}, \underline{\sigma}) = \frac{1}{2} \langle \underline{A}\underline{v}, \underline{v} \rangle - \frac{1}{2} \langle \underline{\sigma}, \underline{T}\underline{\sigma} \rangle + \langle \underline{R}^-\underline{q}, \underline{v} \rangle + \langle \underline{R}^+\underline{\sigma}, \underline{v} \rangle - \Phi_{0\epsilon}(\underline{\sigma}) \quad (3.3)$$

defined on the convex set $V \times K'$, i.e.:

$$\begin{aligned} \delta F_0(\underline{v}, \underline{\sigma}; \delta \underline{v}, \delta \underline{\sigma} - \underline{\sigma}) &\leq 0, \\ V(\delta \underline{v}, \delta \underline{\sigma}) &\in V \times K'. \end{aligned}$$

In ASCIONE and BRUNO [1985a] the existence of a solution of the problem (2.8) is proven under the following compatibility condition on the loads \underline{q} :

$$\exists \underline{\sigma} \in K' : \langle \underline{R}^+\underline{\sigma} - \underline{R}^-\underline{q}, \underline{v}^R \rangle = 0, \quad (3.5)$$

for each plate rigid displacement \underline{v}^R .

In the present paper we recall the main features of the proof because it suggests a simple iterative algorithm for the numerical calculus of the solution. The fundamental points of the proof are the following:

- i) An arbitrary function $\tau_0 > 0$ is assigned,
- ii) A unique solution $(\underline{v}_0, \underline{\sigma}_0)$ of the auxiliary problem, corresponding to τ_0 , exists,
- iii) $fJ\sigma_z^0$ is calculated and we verify if this solution coincides with τ_0 . In the affirmative case the couple $(\underline{v}_0, \underline{\sigma}_0)$ is a solution of the problem (2.7), otherwise we go back to point ii) and we substitute τ_0 with $fJ\sigma_z^0$. In this way an iterative procedure corresponding to point i) and iii) is recognized,
- iv) Evidently point iii) leads to a fixed point problem for the operator:

$$\underline{B} : \tau_0 > 0 \rightarrow fJ\sigma_z^0 \quad (3.6)$$

It can be show that at least one element τ^* exists such that $\underline{B}(\tau^*) = \tau^*$ and evidently the solution of the auxiliary problem corresponding to τ^* is a solution of the problem (2.8).

Moreover, operator \underline{B} becomes a contraction mapping for a sufficiently small friction coefficient and then we can obtain a result of uniqueness, too [13].

The numerical difficulties related to the constraint $\underline{\sigma} \in K'$ in problem (3.4) can be overcome by the penalty approach which follows below. We define the exterior penalty functional:

$$P(\underline{\sigma}) = \frac{1}{2} \|\sigma_z^-\|_{V_f'} \quad (3.7)$$

where σ_z^- is the negative part of σ_z , defined as the projection of σ_z on the negative cone of V_f' , and consider the penalized functional:

$$F_{0\eta}(\underline{v}, \underline{\sigma}) = F_0(\underline{v}, \underline{\sigma}) - \frac{1}{\eta} P(\underline{\sigma}) \quad (3.8)$$

Approximate solutions of problem (3.4) are obtained by finding [13] the stationarity points of the saddle functional $F_{0\eta}$ in the whole space $V \times V_f'$, i.e. the points satisfying the variational equation:

$$\delta F_{0\eta}(\underline{v}_\eta, \underline{\sigma}_\eta; \delta \underline{v}, \delta \underline{\sigma}) = 0 \quad \forall (\delta \underline{v}, \delta \underline{\sigma}) \in V \times V_f' \quad (3.9)$$

This approximate problem admits a unique solution $(\underline{v}_\eta, \underline{\sigma}_\eta)$ under the load compatibility condition (3.5) and the family of penalized solution $\{(\underline{v}_\eta, \underline{\sigma}_\eta)\}$ converges strongly [13] to the solution of problem (3.4) when $\eta \rightarrow 0$, i.e.:

$$\lim \|\underline{v}_{0\eta} - \underline{v}_0\|_V = \lim \|\underline{\sigma}_{0\eta} - \underline{\sigma}_0\|_{V_f'} = 0 \quad (3.10)$$

4. Applications

In this section we utilize the variational formulation previously introduced to analyze, via finite elements, the contact problems shown in Figs. 4.

In both problems the unknown functions $u_s, w, \psi_x, \sigma_x, \sigma_z$ are only dependent on the coordinate x .

The finite element approximation are put in the form of:

$$u_s^h = u_{s\alpha} f_\alpha(x), \quad (4.1a)$$

$$w^h = w_\alpha f_\alpha(x), \quad (4.1b)$$

$$\psi_x^h = \psi_{x\alpha} f_\alpha(x), \quad (4.1c)$$

$$\sigma_x^h = \sigma_{x\beta} g_\beta(x), \quad (4.1d)$$

$$\sigma_z^h = \sigma_{z\beta} g_\beta(x), \quad (4.1e)$$

where f_α and g_β are the global interpolating functions and α, β are dummy indices. Moreover, we assume that the datum τ_0 is of the type:

$$\tau_0(x) = \tau_{x\beta^{(0)}} g_\beta(x). \quad (4.1f)$$

By means of easy substitutions we achieve the following discrete of stationary condition (3.9):

$$\underline{K}\underline{v} - \underline{q} + \underline{M}^T \underline{r} = \underline{0}, \quad (4.2a)$$

$$- \underline{T}\underline{r} + \underline{M}\underline{v} - \epsilon \underline{S}_0 \underline{r} +$$

$$- \frac{1-\epsilon^2}{\epsilon} \underline{Q}(\underline{r}) - \frac{1}{\eta} \underline{P}(\underline{r}) = \underline{0}. \quad (4.2b)$$

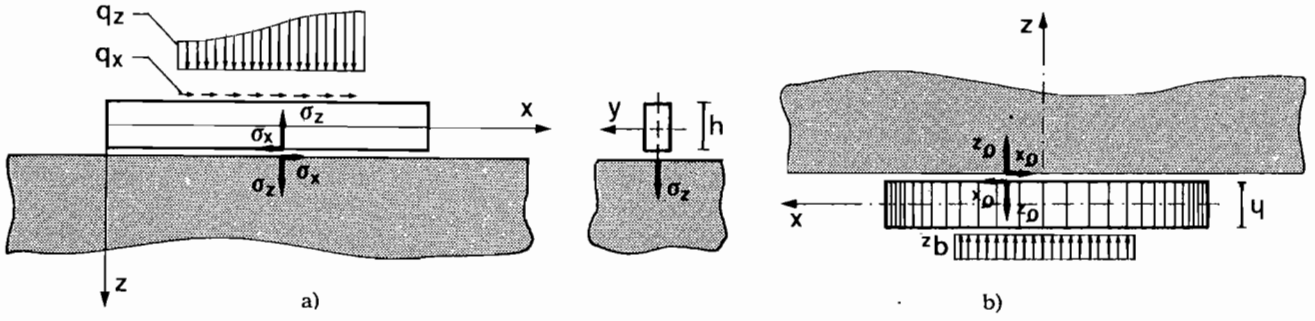


Fig. 4

In eqs (4.2) the matrices \underline{K} , \underline{M} , \underline{T} , \underline{S}_o , \underline{q} are classically defined by the usual finite element techniques.

In regards the non linear terms $\underline{Q}(\underline{r})$ and $\underline{P}(\underline{r})$, which can be written in the block form:

$$\underline{Q}(\underline{r}) = \begin{bmatrix} \underline{Q}_x(\underline{r}_x) \\ \underline{Q} \end{bmatrix}, \quad (4.3)$$

$$\underline{P}(\underline{r}) = \begin{bmatrix} \underline{Q} \\ \underline{P}_z(\underline{r}_z) \end{bmatrix},$$

we obtain:

$$\underline{Q}_{x\beta}(\underline{r}_x) = \int_{\Omega} [(|\sigma_x^h| - \tau_o)^+ \frac{\sigma_x^h}{|\sigma_x^h|} g_{\beta}] d\Omega =$$

$$= \int_{\Omega} [(|\sigma_{x\alpha} g_{\alpha}| - \tau_o)^+ \frac{\sigma_{x\alpha} g_{\alpha}}{|\sigma_{x\alpha} g_{\alpha}|} g_{\beta}] d\Omega, \quad (4.4a)$$

$$\bar{W}_{\gamma e} = \begin{cases} 0 & \text{if } |\sigma_{x\beta} g_{\beta}(\xi_{\gamma e})| \leq \tau_o(\xi_{\gamma e}), \\ \text{Gaussian weight at the point } \xi_{\gamma e} & \text{if } |\sigma_{x\beta} g_{\beta}(\xi_{\gamma e})| > \tau_o(\xi_{\gamma e}), \end{cases} \quad (4.7a)$$

being

$$W_{\gamma e} = \begin{cases} 0 & \text{if } \sigma_{z\beta} g_{\beta}(\xi_{\gamma e}) > 0 \\ \text{Gaussian weight at the point } \xi_{\gamma e} & \text{if } \sigma_{z\beta} g_{\beta}(\xi_{\gamma e}) \leq 0 \end{cases} \quad (4.7b)$$

In eqs (4.6) the sums are extended to all N_e finite elements and Γ_e gaussian points on the e -th element.

Moreover, in eq. (4.5a) the vector $\underline{r}_x^{(o)}$ is defined as:

$$\underline{r}_{x\alpha}^{(o)} = \underline{r}_{x\alpha} \frac{\Gamma_{x\alpha}}{|\Gamma_{x\alpha}|} \quad (4.8)$$

Relations (4.7) and (4.8) clary, in the light of what has been presented in § 3, the algorithm utilized in the numerical examples, which will be shown in the next section.

$$\underline{P}_{\beta}(\underline{r}_z) = \int_{\Omega} (\sigma_z^h)^- g_{\beta} d\Omega = \int_{\Omega} (\sigma_{z\alpha} g_{\alpha})^- g_{\beta} d\Omega. \quad (4.4b)$$

In eqs (4.4) the integrals can be numerically evaluated by means of Gauss' quadrature; it is convenient to put vectors \underline{Q}_x and \underline{P}_z in the form:

$$\underline{Q}_x(\underline{r}_x) = \bar{\underline{G}}(\underline{r}_x) \underline{r}_x - \bar{\underline{G}}(\underline{r}_x) \underline{r}_x^{(o)}, \quad (4.5a)$$

$$\underline{P}_z(\underline{r}_z) = \underline{\underline{G}}(\underline{r}_z) \underline{r}_z, \quad (4.5b)$$

where the matrices $\bar{\underline{G}}$ and $\underline{\underline{G}}$ are defined by the following relations:

$$\bar{\underline{G}}_{\alpha\beta} = \sum_e \sum_{\gamma e} \bar{W}_{\gamma e} g_{\alpha}(\xi_{\gamma e}) g_{\beta}(\xi_{\gamma e}), \quad (4.6a)$$

$$\underline{\underline{G}}_{\alpha\beta} = \sum_e \sum_{\gamma e} W_{\gamma e} g_{\alpha}(\xi_{\gamma e}) g_{\beta}(\xi_{\gamma e}), \quad (4.6b)$$

In particular, within each auxiliary problem (4.2), the positions (4.6) and (4.8) enable us to adopt a simple iterative procedure in which, at each step, the quantities $\bar{W}_{\gamma e}$, $W_{\gamma e}$ and $\Gamma_{x\alpha}/|\Gamma_{x\alpha}|$ are evaluated on the basis of the results obtained in the previous step.

5. Numerical results

In this section we present some numerical results relative to the one-dimensional models of beam and axisymmetric circular plate subject to the load conditions shown in Fig. 5.

In order to deal with nondimensional quantities, we put:

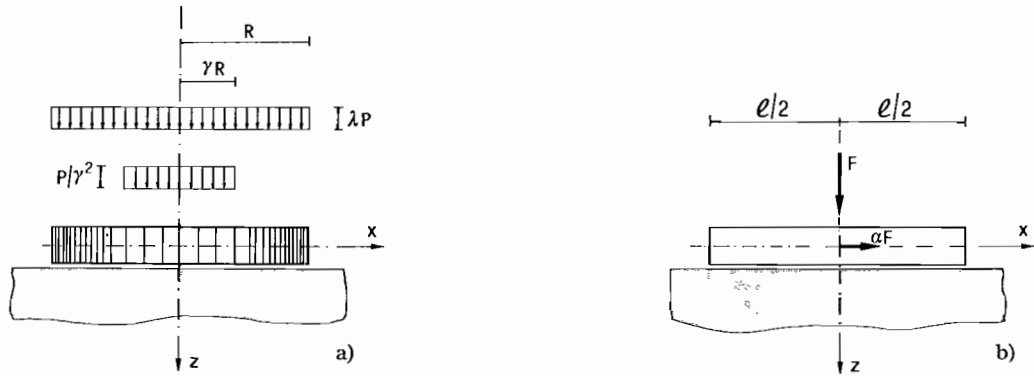


Fig. 5

Beam

$$t = \frac{x}{l} \quad t \in \left[-\frac{1}{2}, \frac{1}{2} \right],$$

$$U = \frac{D}{Fl^3} u,$$

$$W = \frac{D}{Fl^3} w,$$

$$\Psi = \frac{D}{Fl^2} \psi,$$

$$\Sigma_x = \frac{l}{F} \sigma_x,$$

$$\Sigma_z = \frac{l}{F} \sigma_z,$$

Circular plate

$$t = \frac{x}{l} \quad t \in [0, 1], \quad (5.1a)$$

$$U = \frac{D}{Fl^2} u, \quad (5.1b)$$

$$W = \frac{D}{Fl^2} w, \quad (5.1c)$$

$$\Psi = \frac{D}{Fl} \psi, \quad (5.1d)$$

$$\Sigma_x = \frac{\pi l^2}{F} \sigma_x, \quad (5.1e)$$

$$\Sigma_z = \frac{\pi l^2}{F} \sigma_z, \quad (5.1f)$$

It is easy to show that the solution of the problem depend on the following dimensionless parameters:

$$\zeta = \frac{h}{l}, \quad \zeta = \frac{h}{l}, \quad (5.2a)$$

$$\rho = \frac{32D(1-\nu_f^2)}{\pi E_f l^4}, \quad \rho = \frac{2D(1-\nu_f^2)}{E_f l^3}, \quad (5.2b)$$

$$\nu, \nu_f \quad \nu, \nu_f, \quad (5.2c)$$

$$f, \quad f, \quad (5.2d)$$

$$\bar{\epsilon} = 4\pi G_f \epsilon, \quad \bar{\epsilon} = \frac{4\pi G_f}{l} \epsilon, \quad (5.2e)$$

of which the numerical analyzed examples show the influence.

In all the examined cases we have assumed $\nu = 0.3$ and $\nu_f = 0.1$.

5.1 Beam example

Firstly, we investigate on the convergence of the finite element numerical model.

A first group of results (Fig. 6) concerns the case of a uniform mesh of N_e elements, over which the unknown functions U , W , ψ , Σ_x and Σ_z are approximated by linear polynomials.

The results of Fig. 6 refer to a beam more flexible compared to the elastic half-space ($\rho = 0.00025$) with a contact region equal to about 0.4 of complete length.

We observe a good convergence of the numerical model especially as far as the displacements are concerned while as far as the interaction Σ_x and Σ_z are concerned the convergence appears slower. This behaviour is justified by the fact that the above mentioned interactions are approximated by means of a lower number of finite elements due to the loss of contact between beam and foundation.

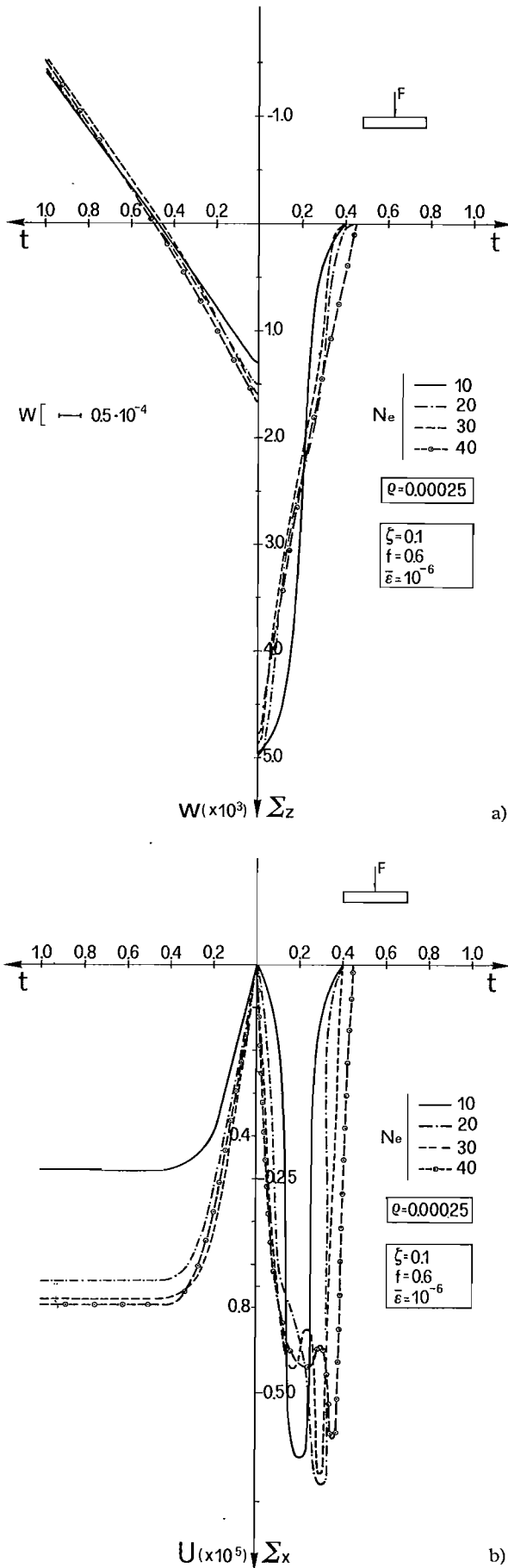


Fig. 6

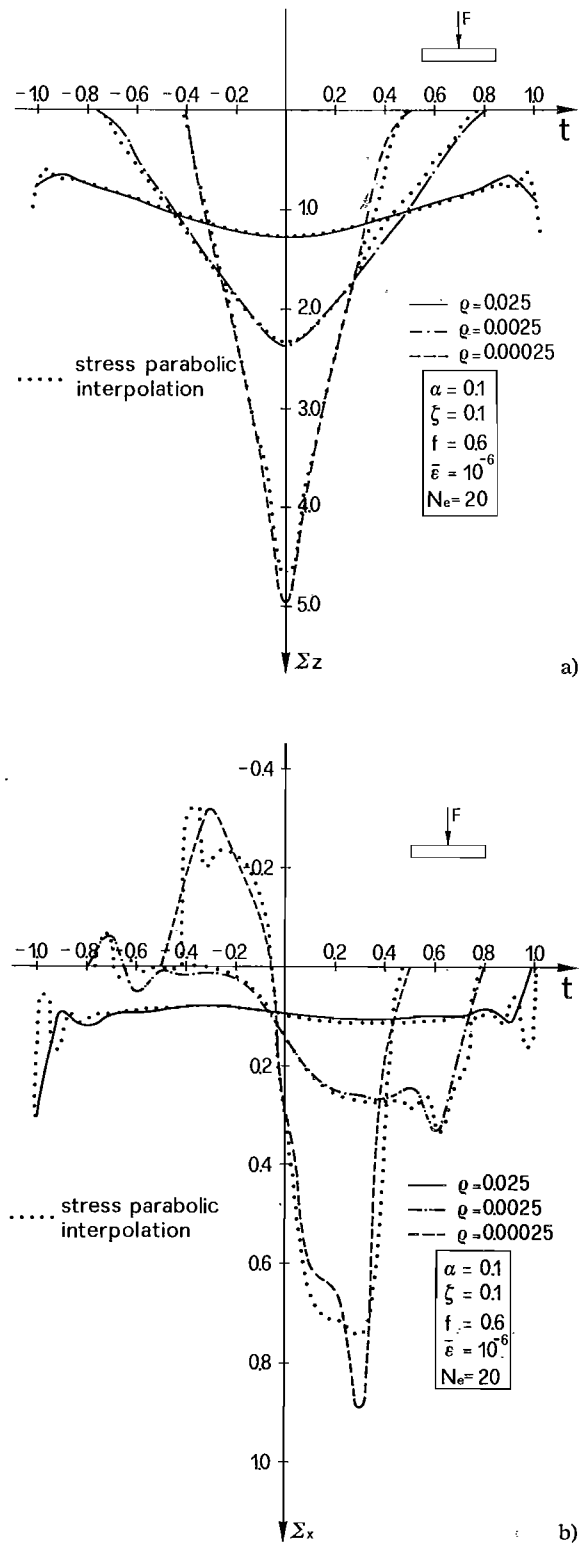


Fig. 7

A second group of results (Fig. 7) is relative to an approximation by linear polynomials for W and ψ and quadratic ones for Σ_x and Σ_z .

The mesh is made up of 20 finite elements of equal length and over each element the parabolic approximation of the interactions is of the type shown in Fig. 8 which assures the con-

tinuity of the first derivative at the end nodes of the element.

The investigation is carried out for several values of the relative stiffness parameter ρ and the obtained results are compared to the ones relative to the linear interpolation of all unknown functions.

A third group of results, finally, compares (Fig. 9) different approximations of displacement functions U , W , and ψ . The choice made are the following: a) linear approximations of all three functions ($N_e = 30$); b) linear approximations of U , ψ and the parabolic ones (Fig. 8)

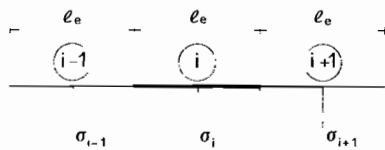


Fig. 8

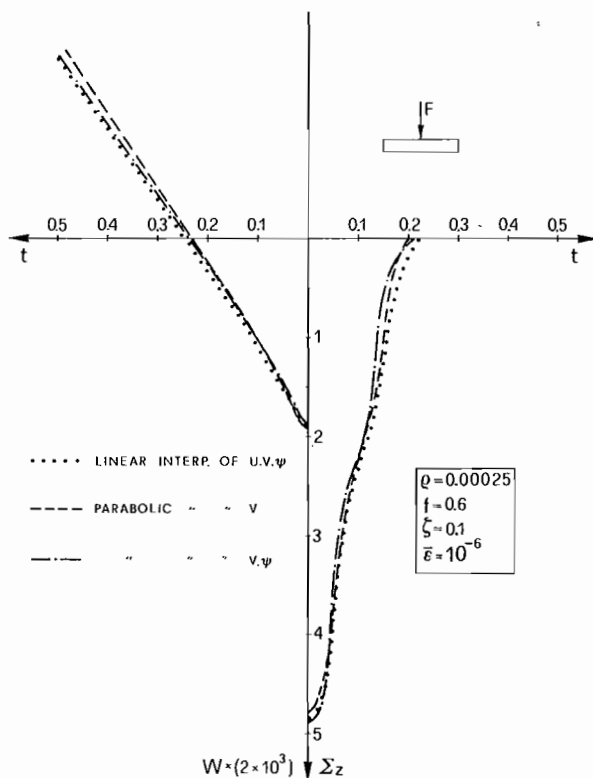


Fig. 9

of W ($N_e = 18$); c) linear approximations of U , the parabolic ones of W and ψ ($N_e = 22$).

In all cases the interactions Σ_x and Σ_z are interpolated by means of linear functions. The choice of the element number has been made

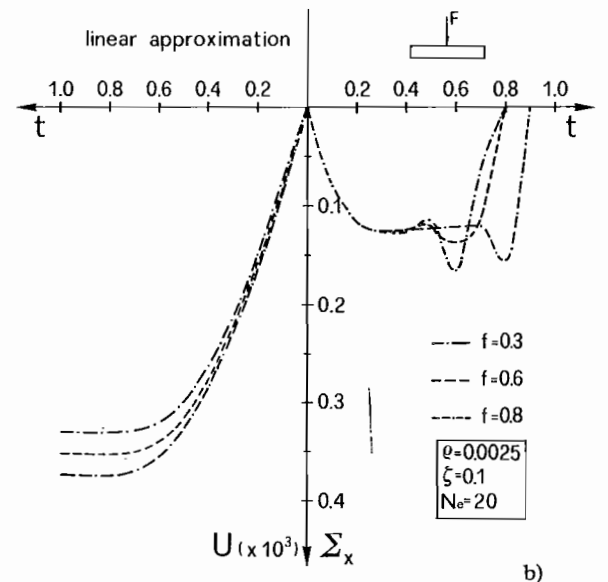
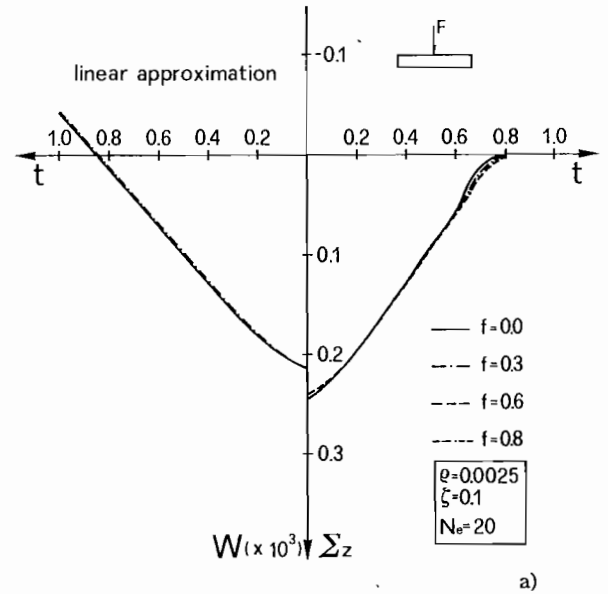


Fig. 10

by keeping constant the overall number of freedom degrees of the unknowns displacement (93, 93, 91).

Subsequently, we have investigated the influence of the parameter $\bar{\epsilon}$ on the solution. For this reason Tab. 1, in which the values of W ,

TABLE 1
LINEAR APPROXIMATION OF ALL VARIABLES

$\bar{\epsilon}$	W	Σ_x	Σ_z
10^{-1}	0.21138×10^{-2}	0.13397	2.3934
10^{-4}	0.21235×10^{-2}	0.14348	2.4081
10^{-7}	0.21235×10^{-2}	0.14348	2.4082

($\rho = 0.0025$; $f = 0.6$; $\alpha = 0.1$; $N_e = 20$)

Σ_x and Σ_z relative to the middle section are carried over for the different choices of $\bar{\epsilon}$, is significative.

We observe that for values of $\epsilon < 10^{-4}$ the solution is practically independent from this parameter and it can be assumed as approximation of the contact problem with Coulomb's friction law.

A similar behaviour with respect to ϵ has been found also in the case of the plate example. Therefore, in the next numerical applications we will assume $\bar{\epsilon} = 10^{-6}$.

Finally, Fig. 10 shows the influence of the friction coefficient f on the solution. We observe that for the beam example the solution is less appreciable to the variations of this coefficient.

5.2 Plate example

The numerical results for the plate example refer to the load condition shown in Fig. 5. First of all for this load condition the model behaviour is analyzed in function to the relative stiffness parameter ρ between plate and elastic half-space. Figs. 11 to 18 shows this behaviour; we observe the remarkable influence of this parameter both on the displacements U and W

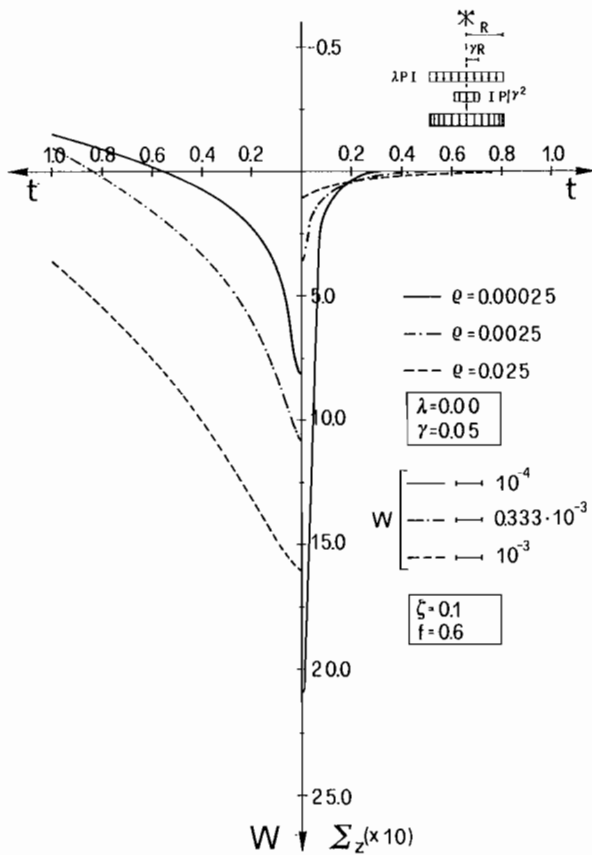


Fig. 11

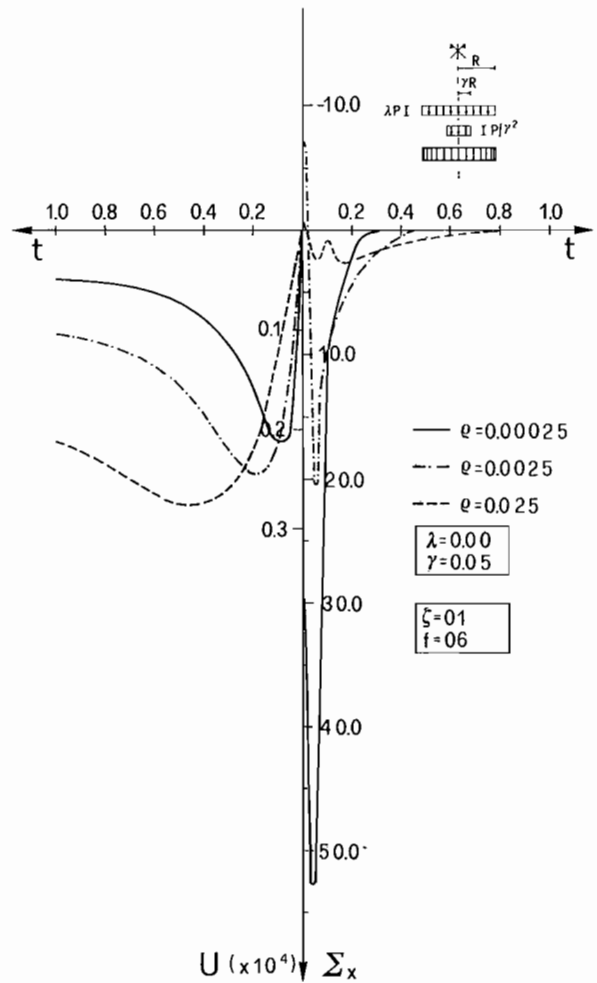


Fig. 12

and on the stresses Σ_z and Σ_x . Obviously such influence is reflected in a substantial modification of the contact region between plate and elastic half-space.

In Fig. 19 the influence of the friction coefficient f is analyzed. Contrarily of the beam example in this case a not negligible influence of this parameter both on displacements and on interactions is pointed out.

Finally the effects of the tangential stresses, which arise on the plate-halfspace interface, on the plate displacements and on the half-space stresses are analyzed. These effects are connected to the work which the above mentioned stresses accomplish for the horizontal plate displacements on the half-space interface.

The numerical investigation is developed for fixed values of the friction coefficient f and of the relative stiffness parameter ρ . It shows (Fig. 20) as a parameter $\zeta = h/l$ can modify the value of the displacements and that of the interactions exerting a remarkable influence comparable to ρ one.

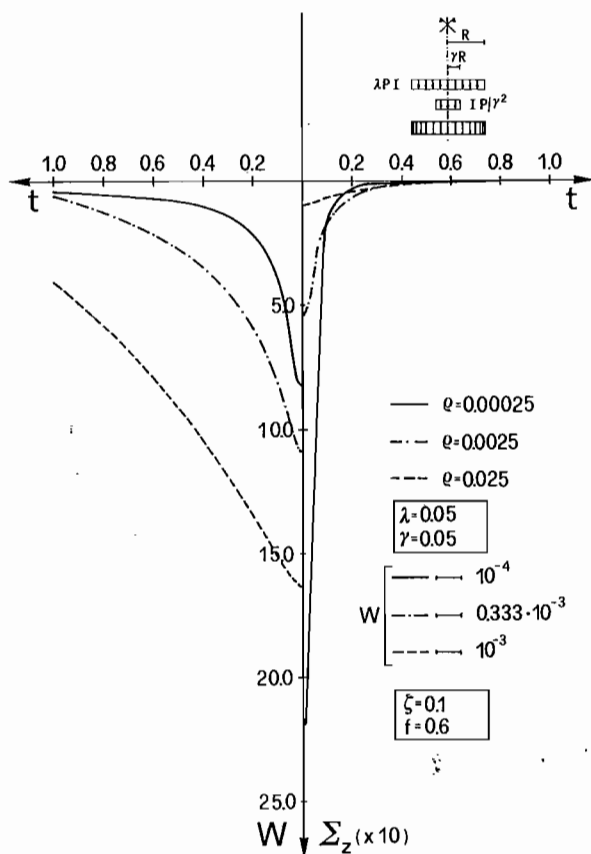


Fig. 13

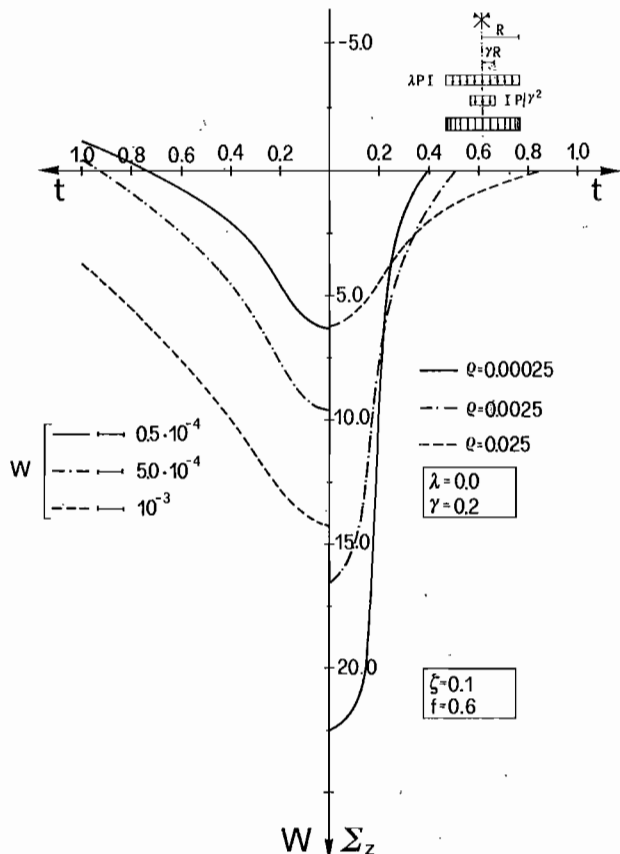


Fig. 15

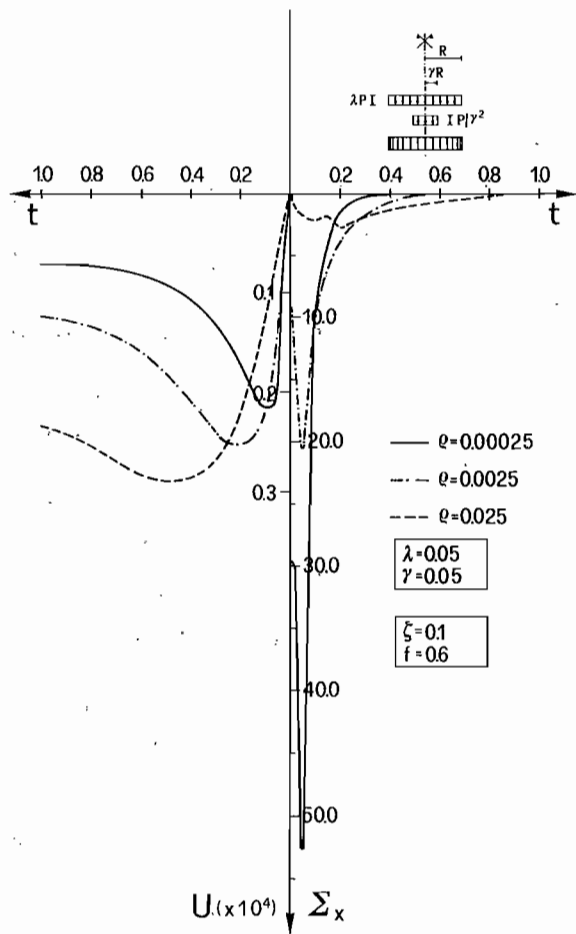


Fig. 14

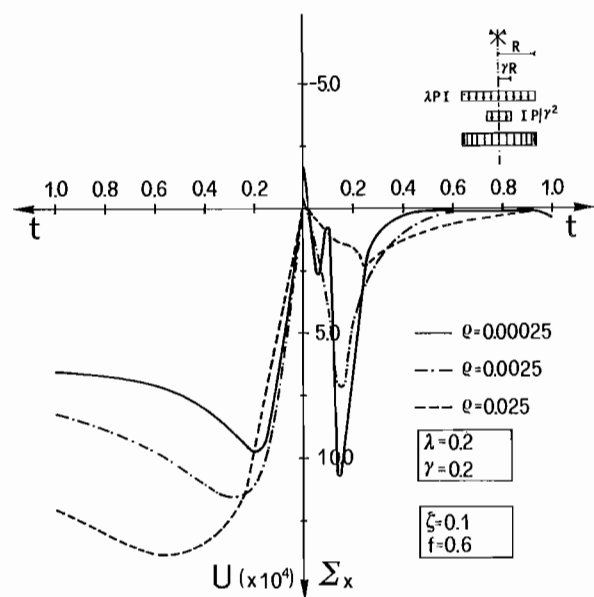


Fig. 16

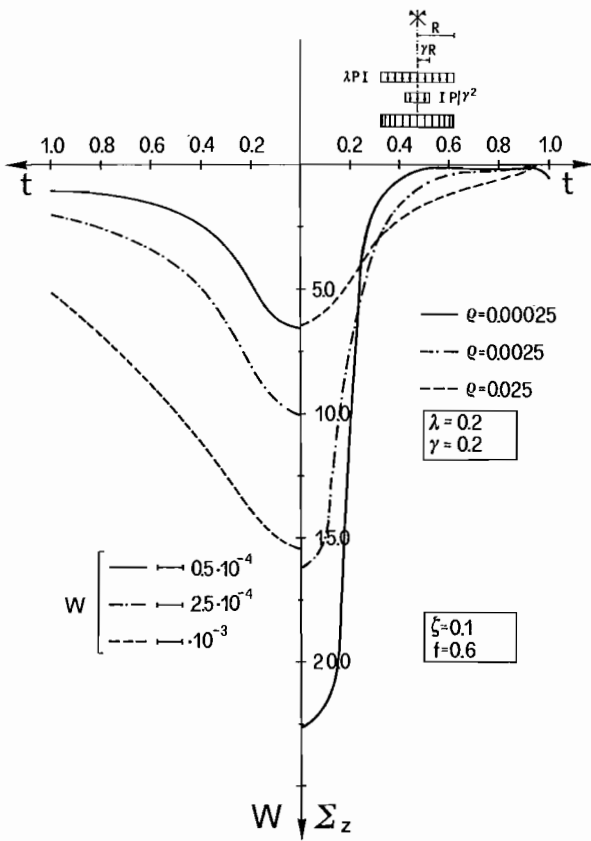


Fig. 17

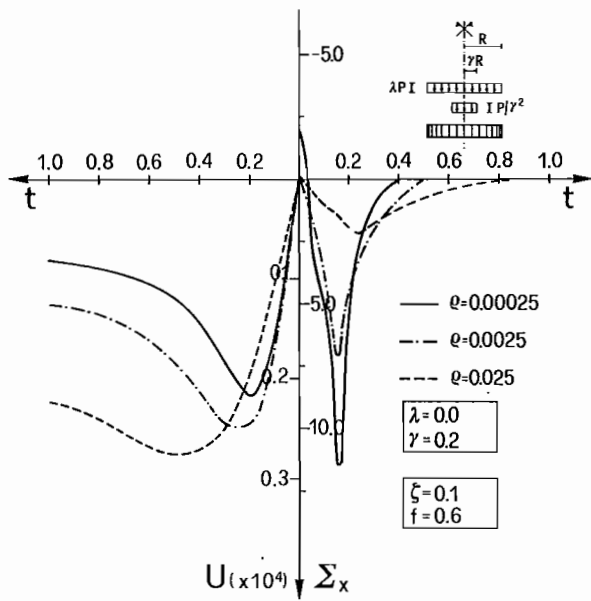
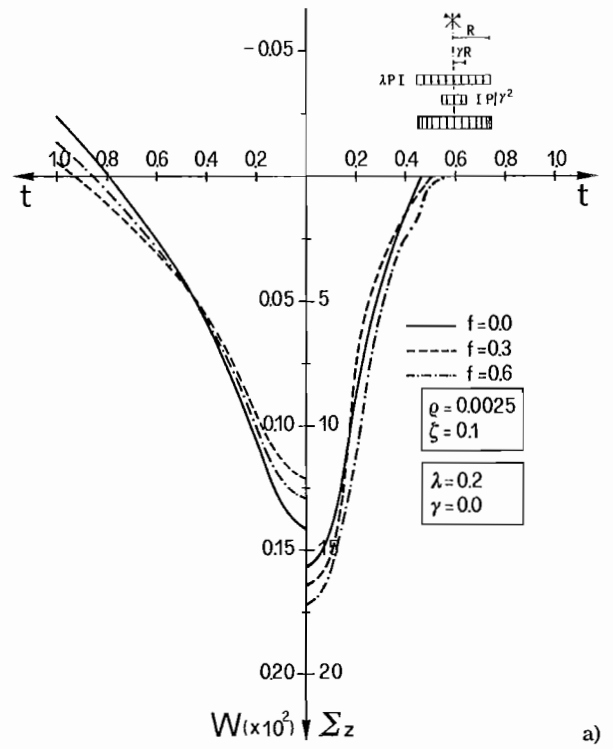


Fig. 18

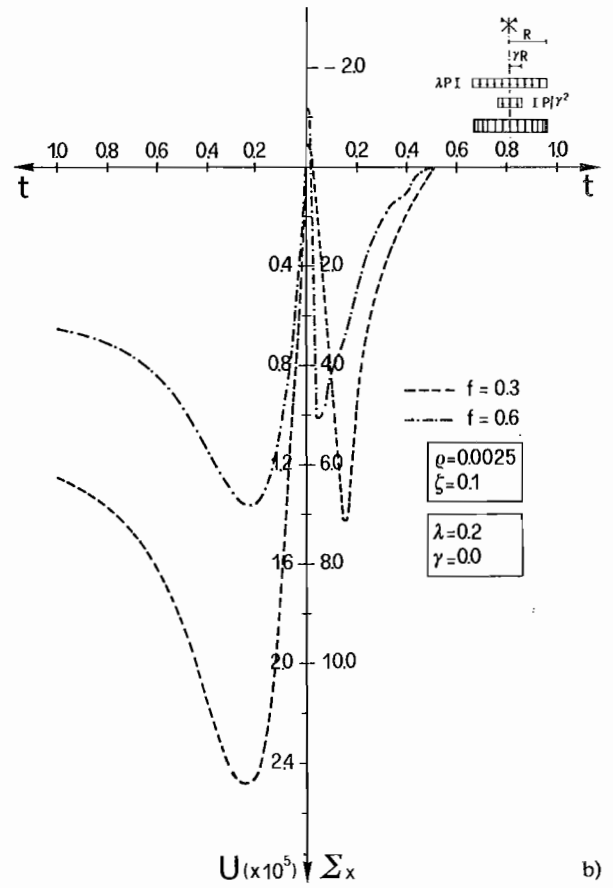


Fig. 19

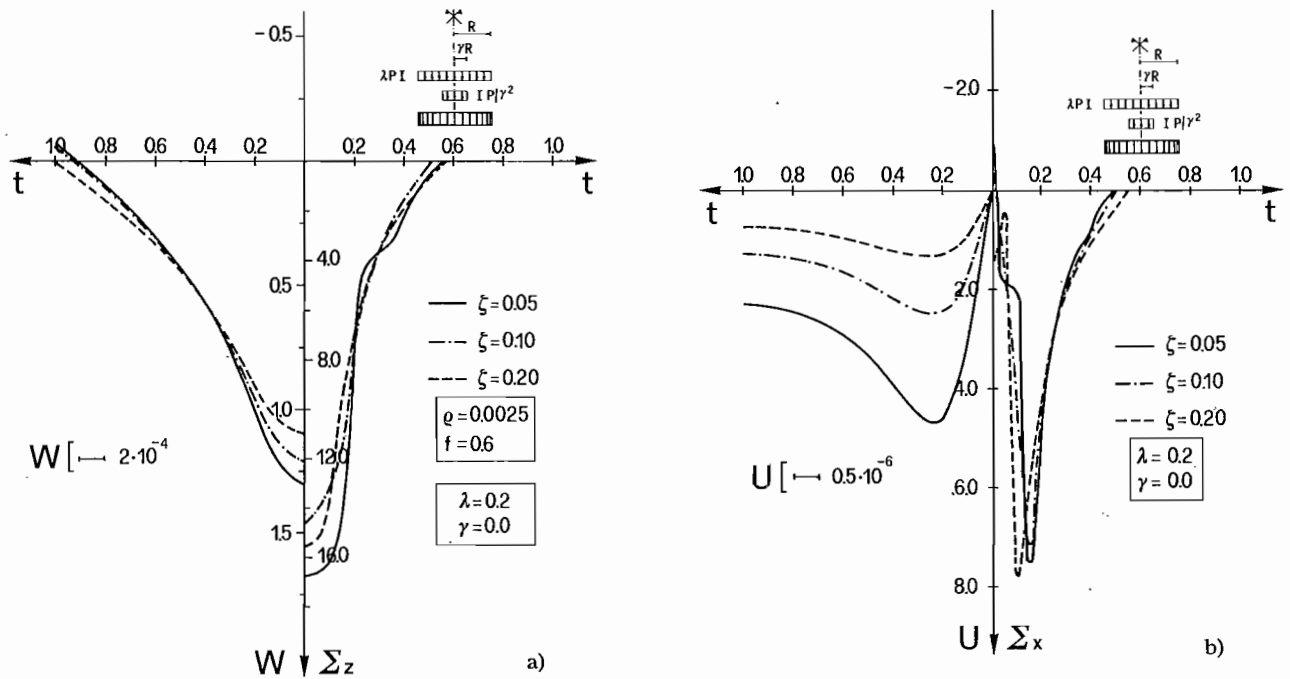


Fig. 20

SYMBOLS

- Ω : open region corresponding to the plate middle plane,
- $\underline{u} = (u_x, u_y, u_z)$: plate displacement components,
- $\underline{q} = (q_x, q_y, q_z)$: load components acting on the upper surface Ω^- of the plate,
- $\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$: interaction components between the lower surface Ω^+ of the plate,
- $\underline{\tau} = (\tau_x, \tau_y, 0)$: shear components of $\underline{\sigma}$,
- $\underline{u}_f = (u_f, v_f, w_f)$: displacement components of the free surface of the foundation,
- $\underline{\delta} = (\delta_x, \delta_y, \delta_z)$: relative displacement components between plate and foundation,
- $\underline{\delta} = (\delta_x, \delta_y, 0)$: shear components of $\underline{\delta}$,
- w : vertical displacement of the plate middle plane,
- ψ_x : plate flexural bending slope along the x axis,
- ψ_y : plate flexural bending slope along the y axis,
- f : friction coefficient,
- u_s, v_s : stretching displacements of the plate middle plane,
- ϵ : regularization parameter in the friction law,
- C : stretching stiffness,
- k : shear correction factor,
- ν : Poisson's ratio of the plate (beam),
- ν_f : Poisson's ratio of the elastic half-space,
- G : shear elasticity modulus of the plate (beam),
- G_f : shear elasticity modulus of the elastic half-space.

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Indagine numerica sul problema di contatto unilaterale con attrito

Negli ultimi anni i problemi di contatto unilaterale hanno attirato l'attenzione di numerosi ricercatori, che hanno sviluppato un'ampia bibliografia sull'argomento con interessi variegati dalla Meccanica dei Solidi all'Ingegneria Geotecnica.

In quest'ambito, di particolare interesse tecnico per le applicazioni di carattere ingegneristico sono i problemi di contatto di travi e piastre su fondazioni elastiche.

È ben noto che la risposta di un terreno non può mai essere schematizzata come perfettamente elastica; tuttavia tale ipotesi è comunemente riconosciuta come valida per la valutazione delle interazioni suolo-struttura, che sono di rilevante interesse sia per le verifiche di sicurezza delle strutture che per la previsione dei cedimenti del terreno.

Nello studio di tali interazioni può in alcuni casi essere necessario tener conto dell'incapacità del terreno a resistere a sforzi di trazione con la conseguente possibile creazione di zone di distacco tra struttura e fondazione. Tale situazione può prodursi sia per una flessibilità relativamente (al terreno) forte della struttura in esame, sia per la concorrenza di particolari distribuzioni di carico, come ad esempio quelle derivanti da eventi sismici.

Quando le zone di distacco diventano percentualmente significative, l'ipotesi usuale di contatto bilaterale o adesivo può condurre a valutazioni errate sia per lo strutturista che per il geotecnico. Di qui l'interesse ad introdurre nella teoria elastica l'ipotesi di contatto monolaterale con o senza attrito.

I problemi di contatto monolaterali senza attrito sono stati ampiamente sviluppati nell'ultimo decennio sia sotto il profilo teorico che numerico.

Meno investigati sono invece i problemi di contatto in presenza di attrito, che può assumere un'importanza rilevante quando la zona di effettivo contatto tra struttura e fondazione tende a ridursi.

A differenza dei problemi di contatto senza attrito, questi ultimi presentano infatti notevolissime difficoltà analitiche, soprattutto quando la legge di attrito sia espressa nella forma classica proposta da Coulomb [DUVAUT, 1980]. Ciò giustifica lo sforzo di attribuire a questa legge delle formulazioni alternative in grado di modellare con altrettanta semplicità ed aderenza l'effettivo fenomeno fisico dell'attrito, consentendo altresì di superare le difficoltà matematiche sopra richiamate. A questa classe appartengono ad esempio le leggi di tipo non locale proposte da ODEN *et al.* [1981, 1982] e quella più recentemente indicata da ASCIONE e BRUNO [1985a, b] corrispondente ad una regolarizzazione della legge di Coulomb. Tale legge è illustrata per un caso monodimensionale in Fig. 2, da cui si evince che lo slittamento relativo prevedibile all'interfaccia tra struttura e fondazione è sempre diverso da zero anche se la sua entità diviene rilevante solo quando il valore della tensione tangenziale supera il limite $f\sigma_x$. Ciò è in contrasto con la legge di Coulomb ma è pienamente giustificabile sul piano fisico, in quanto effettivamente lo spostamento relativo si manifesta già per valori bassi della tensione tangenziale, essendo legato alla deformazione elasto-plastica delle asperità presenti sulle superfici a contatto: è solo quando la resistenza offerta, da queste asperità nei confronti delle deformazioni, viene completamente vinta che si instaura il fenomeno macroscopico dello scorrimento, cosiddetto da superamento dell'attrito.

Il lavoro in oggetto si propone di svolgere un'indagine numerica sul problema di contatto monolaterale con attrito di travi e piastre su un semispazio elastico isotropo ed omogeneo, adottando come legge di attrito quella proposta da Ascione e Bruno.

Il modello cinematico assunto per la piastra (o per la

trave) è quello suggerito da Mindlin, corrispondente alle espressioni (2.1) delle componenti di spostamento. In esse, accanto allo spostamento trasversale w e alle componenti estensionali u_x e v_x , compaiono come variabili indipendenti le rotazioni flessionali ψ_x e ψ_y lungo le direzioni degli assi coordinati. Tali variabili prendono il posto delle derivate parziali $\partial w/\partial x$ e $\partial w/\partial y$ della classica teoria delle piastre sottili. In tal modo il modello offre la possibilità di tener conto dell'effetto delle tensioni tangenziali sulla deformazione, fenomeno che può risultare rilevante in strutture tozze come tipicamente sono le strutture di fondazione. Un'immediata conseguenza della (2.1) è che le corde inizialmente ortogonali al piano medio della piastra restano in seguito alla deformazione ancora rettilinee (assenza di ingobbamento) ma non risultano più ortogonali alla superficie media.

In particolare il lavoro si limita, almeno in questa prima fase, a prendere in esame solo modelli monodimensionali corrispondenti agli esempi di travi o piastre circolari assialsimmetriche nelle condizioni di carico illustrate in Fig. 5.

La formulazione adottata utilizza come incognite sia i parametri di spostamento della piastra che le interazioni tra struttura e semispazio elastico. Tale formulazione si presenta infatti particolarmente vantaggiosa per le applicazioni numeriche. In ordine ai problemi teorici connessi con questioni di esistenza, unicità e approssimazione della soluzione si rinvia ad ASCIONE e BRUNO [1985a].

L'aspetto messo in risalto in questo lavoro è essenzialmente quello numerico attraverso un approccio agli elementi finiti.

Il problema affrontato è tipicamente un problema non lineare per la presenza del vincolo $\sigma_x \geq 0$ legato all'ipotesi di incapacità della fondazione a resistere a sforzi di trazione (Fig. 4) ed alla legge di attrito: si ricordi la Fig. 2 ed il limite $f\sigma_x$ che dipende dalla soluzione e come tale non è noto a priori.

Esso viene quindi risolto mediante un procedimento iterativo organizzato in due cicli di iterazioni, l'uno interno all'altro. Nel ciclo più esterno si risolve una successione di problemi ausiliari ottenuti attribuendo al limite $f\sigma_x$ il valore desunto dal passo precedente (si parte da un valore iniziale τ_0 (Fig. 3)). Nel ciclo più interno si affronta invece il trattamento numerico del vincolo $\sigma_x \geq 0$ mediante una tecnica di tipo «penalty». Dal punto di vista meccanico essa si traduce nel partire da una situazione di contatto adesivo e risolvere quindi una successione di ulteriori schemi ausiliari nei quali, tra piastra e semispazio, nelle zone dove al passo precedente si erano registrate delle interazioni di trazione ($\sigma_x < 0$), sono interposte delle molle di rigidità $\eta (\rightarrow 0)$.

Entrambi i cicli sono arrestati quando la differenza tra due soluzioni successive è minore di una tolleranza prestabilita.

Per quanto concerne l'esempio trave è stata svolta, innanzitutto, un'indagine sulla convergenza del modello numerico agli elementi finiti. Un primo gruppo di risultati (Fig. 6) riguarda il caso di una mesh uniforme di N_e elementi sui quali le funzioni incognite U , W , ψ , Σ_x e Σ_y (adimensionalizzazione di u_x , w , ψ_x , σ_x e σ_y definite nelle (5.1)) sono interpolate mediante polinomi di primo grado.

Tali risultati si riferiscono ad una trave molto flessibile rispetto al semispazio elastico ($\rho = 0.00025$) con una zona di contatto pari a circa il 40% dell'intera lunghezza. Si osserva una buona convergenza del modello numerico soprattutto per quanto riguarda gli spostamenti, mentre per quanto riguarda le interazioni Σ_x e Σ_y la convergenza risulta più lenta. Tale comportamento può essere giustificato dal fatto che le suddette interazioni sono approssimate mediante un numero minore di elementi finiti a causa del distacco tra la trave e la fondazione.

La Fig. 7 mostra i risultati relativi ad una diversa scelta delle funzioni interpolanti e precisamente polinomi di pri-

mo grado per U , W e ψ e polinomi di secondo grado per Σ_x e Σ_z .

L'approssimazione parabolica delle interazioni è del tipo indicato in Fig. 8, che garantisce la continuità della derivata prima sui nodi estremi dell'elemento.

L'indagine è stata svolta per diversi valori del parametro adimensionale di rigidezza relativa ρ tra piastra e semispazio elastico definito in (5.2b)) ed i risultati ottenuti sono confrontati con quelli relativi all'interpolazione lineare di tutte le funzioni incognite.

Infine, un terzo gruppo di risultati mette a confronto diverse approssimazioni delle funzioni di spostamento U , W e ψ , mentre le interazioni Σ_x e Σ_z sono interpolate mediante funzioni lineari. La scelta del numero di elementi è stata fatta tenendo costante il numero complessivo di gradi di libertà delle incognite di spostamento.

Successivamente si è indagata l'influenza del parametro ϵ sulla soluzione ed i relativi risultati sono riportati in Tab. 1. Si osserva che per valori di $\bar{\epsilon} < 10^{-4}$ (adimensionalizzazione di ϵ definita in (5.2e)) la soluzione è indipendente da tale parametro con attrito e può essere assunta come approssimazione del problema di contatto con attrito alla Coulomb.

Analogo comportamento si è riscontrato per il modello piastra e pertanto in tutte le relative applicazioni si è assunto il valore $\bar{\epsilon} = 10^{-6}$. L'influenza del coefficiente di attrito f sulla soluzione viene mostrata in Fig. 10; in essa si osserva che la soluzione, per il caso trave, è indipendente da tale parametro.

Per quanto riguarda l'esempio piastra è stato analizzato

innanzitutto il comportamento del modello in funzione del parametro di rigidezza relativa ρ tra piastra e semispazio elastico. Le figg. 11 ÷ 18 mostrano la notevole influenza che tale parametro ha sia sugli spostamenti U e W che sulle tensioni Σ_x e Σ_z . Ovviamente tale influenza si riflette in una modifica sostanziale della zona di contatto tra piastra e semispazio elastico.

La fig. 20 mostra, invece, l'influenza del coefficiente di attrito f . Contrariamente a quanto visto per l'esempio trave, in questo caso si evidenzia una non trascurabile influenza di tale parametro sulla soluzione.

Infine, vengono analizzati gli effetti delle tensioni tangenziali, che insorgono all'interfaccia piastra-semispazio, sugli spostamenti della piastra e sullo stato di sollecitazione sulla superficie del semispazio. Tali effetti sono legati al lavoro che le suddette tensioni compiono per gli spostamenti orizzontali della piastra all'interfaccia con il semispazio.

L'indagine numerica è stata svolta per fissati valori del coefficiente di attrito f e della rigidezza relativa ρ e mostra (Fig. 20) come il parametro $\zeta = h/l$ possa modificare il valore degli spostamenti e delle interazioni con influenza percentuale paragonabile a quella di ρ .

In base ai risultati ottenuti è possibile concludere che l'approccio «penalty» proposto è particolarmente efficiente, consentendo di attingere la soluzione in poche iterazioni. Inoltre resta provato che soprattutto nel caso bidimensionale delle piastre la presenza dell'attrito gioca un ruolo notevole sull'estensione della zona di contatto e quindi sui valori dell'interazione suolo-struttura.