On the implications of the principle of physical causality in the propagation of seismic waves in geomaterials

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Summary

The objective of this paper is to illustrate some important implications of the principle of physical causality in the propagation of seismic waves in geomaterials at low-strain levels. Under these conditions the simplest constitutive model able to capture the capacity exhibited by geomaterials, subjected to small amplitude dynamic excitations, to absorb and dissipate strain energy is linear viscoelasticity. An important result implied by this theory of material behaviour is that the phase velocities of P and S waves and material damping ratio are not independent quantities but they are related by the Kramers-Kronig dispersion equations, which are nothing but a statement of the necessary and sufficient conditions required of a viscoelastic continuum so that a pulse propagating through it satisfies the principle of physical causality. Approximate and recently obtained exact solutions of the Kramers-Kronig equations, which from a mathematical point of view are a pair of linear, singular integral equations with Cauchy kernel, are illustrated. The exact solutions were derived with no simplifying assumptions beyond the so-called *fading memory* hypothesis, a rather weak conjecture well fulfilled by geomaterials which states that the current stress tensor depends more strongly on the recent rather than on the distant strain history. These rigorous solutions of the Kramers-Kronig relations are attractive as they allow, at least in principle, the calculation of frequency-dependent damping ratio from phase velocity dispersion and, inversely, frequency-dependent phase velocity of P and S waves from the spectrum of damping ratio. Thus these solutions conjure up a new approach in determining the small-strain dynamic properties of geomaterials with measurements only of one material function. The theoretical results were validated using experimental data from non-resonant column tests carried out on fine-grained soils. Furthermore, they were shown to be consistent with previously derived approximate solutions of the Kramers-Kronig relations and in particular with the well-known dispersion relation widely used in seismology based on the assumption that material damping ratio is rate-independent over the seismic frequency band.

Keywords: Viscoelasticity, material dispersion, causality, seismic waves, damping ratio, Kramers-Kronig relationships

Introduction

Linear viscoelasticity is the simplest formal theory that can be used to describe the mechanical response of solid, dissipative materials to low-amplitude dynamic excitations. It only requires the validity of a) the small strain assumption, b) the time translation invariance hypothesis and c) the assumption that the current value of the Cauchy stress tensor is assumed to depend solely on the current value of the strain tensor and on the past strain history.

Despite its simplicity this theory has proved to be quite effective in describing phenomena of wave propagation in dissipative materials like soils and rocks at low-strain levels [Pipkin, 1986; Ishihara, 1996; Ben-Menahem and Singh, 2000; Aki and Richards, 2002]. Experimental evidence shows in fact that geomaterials subjected to dynamic excitations exhibit the ability both to store strain energy and to dissipate strain energy over a finite period of time even at very small strain levels, below the *linear cyclic threshold shear strain* [Vucetic, 1994]. Both these phenomena can quite accurately be described by the theory of linear viscoelasticity.

An important corollary predicted by this theory is the functional dependence between the velocity of the propagation of a mechanical disturbance and material attenuation or equivalently damping ratio. This functional coupling is a direct consequence of *dispersion*, a phenomenon by which the phase velocity of a bulk wave propagating in a dissipative medium is frequency-dependent. Material dispersion causes the change of shape of a propagating pulse and it can be shown [Aki and Richards, 2002] that it constitutes a necessary condition to satisfy the fundamental principle of physical *causality*. In mathematical terms the functional dependence between phase velocity and material attenuation of a bulk

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wave propagating in a (linear) viscoelastic medium is expressed by the *Kramers-Kronig* dispersion relations [BRENNAN and STACEY, 1977; TSCHOEGL, 1989; LAKES, 1999] which state that the real and imaginary parts of the corresponding wavenumber form a Hilbert transform pair.

A distinctive feature of the Kramers-Kronig relations is that they establish a relation between two fundamental parameters of a viscoelastic medium allowing the computation of one, say the phase velocity of transverse waves, as a function of the other, say shear damping ratio. This can be very useful since both these parameters and their frequency-dependence are of fundamental importance in applications of geotechnical earthquake engineering and in the current practice in experimental geotechnics they are determined separately using different procedures and often ignoring their frequency-dependence.

The Kramers-Kronig relations are (linear) singular Fredholm integral equations of 2^{nd} kind. Approximate solutions have been proposed by seismologists [AKI and RICHARDS, 2002] by assuming rate-independent (i.e. hysteretic) damping ratio over the seismic bandwidth (i.e. 0.001 – 10 Hz). Although there is experimental evidence that for certain types of geomaterials low-strain, energy dissipation is approximately frequency-independent over the seismic band [SHIBUYA *et al.*, 1995; ABERCROMBIE *et al.,* 1997; AKI and RICHARDS, 2002], this assumption is far from having a general validity as there are several studies that seem to disprove it both in geotechnical engineering especially in clayey soils [ZAVORAL and CAMPANELLA, 1994; LEROUEIL and MARQUES, 1996; D'ONOFRIO *et al.*, 1999; STOKOE *et al.,* 1999; RIX and MENG, 2005] and in seismology [BERCKHEMER *et al.,* 1982; FUKUSHIMA *et al.,* 1992; JACKSON *et al.,* 2002; SATOH, 2006]. Other approximate solutions of the Kramers-Kronig relations were obtained by BOOIJ and THOONE [1982] who assumed that the derivative of the real part of the complex modulus with respect to the logarithm of frequency is constant or varies slowly with frequency. MENG [2003] adopted these simplified solutions to model strain-rate effects of remoulded kaolin and fine-grained natural soils obtaining satisfactory results.

This paper illustrates closed-form, exact solutions of the Kramers-Kronig integral equations. No approximations are involved in their derivation other than those at the base of standard viscoelasticity theory supplemented by the so-called *fading memory* hypothesis, a fairly weak assumption stating that the current state of stress in a viscoelastic body depends more strongly on the recent rather than on the remote strain history. The results obtained constitute a powerful tool for theoretical investigations of strain-rate effects in geomaterials. Also the exact solutions of the Kramers-Kronig dispersion relations will provide a superior and effective tool for the determination of the constitutive parameters of geomaterials and their frequency-dependence laws (i.e. damping ratio and phase velocity spectra). The advantages of the method include a) the possibility of determining the frequency-dependent material damping ratio entirely from phase velocity measurements of P and S waves, b) the possibility conversely of determining the frequency-dependent phase velocity entirely from damping ratio measurements, and c) the fact that these important material functions are obtained by inherently satisfying the fundamental principle of physical causality. Applications of the theoretical results are performed with respect to the approximate solution based on hysteretic damping ratio and experimental data from the literature for fine-grained soils.

Constitutive modelling of linear dissipative continua

Experimental evidence shows that the response of geomaterials to a dynamic excitation is strongly dependent upon the magnitude of the induced cyclic shear strain in uniaxial loading, and of the norm of the deviatoric strain tensor in multiaxial loading. Figure 1 illustrates the concepts of linear ε^t_l and volumetric ε^t_v cyclic threshold shear strains introduced by VUCETIC [1994] in the context of uniaxial loading. For magnitudes of the cyclic strain lower than ε^t_l, soils exhibit a *linear* even though inelastic response. Energy dissipation has a viscous origin and is caused by an interactive combination of several mechanisms whose relative importance depends upon the frequency of exitation, the grain size and the degree of saturation. In Figure 2 the concept of linear and volumetric cyclic threshold shear strains is generalized to the case of multiaxial loading conditions.

As mentioned in the introduction, the simplest, linear constitutive model that can be used to describe the mechanical response of solid, dissipative, materials is linear viscoelasticity. Formulation of the theory of linear viscoelasticity requires essentially three assumptions:

- validity of small strain theory, namely $sup\,|u_{i,j}(\mathbf{x},\tau)| \leq \varepsilon^t_l(\varepsilon_1, \varepsilon_2, \varepsilon_3,\tau)$ with $\tau \in\ ]-\infty, t]$;
- current Cauchy stress tensor $\sigma_{ij}(t)$ depends upon the current value of the strain tensor ε_{ij} and on its past strain history, namely $\sigma_{ij}(t) \leftarrow \Phi$: $\{\varepsilon_{kl}(\tau)\}^t_{-\infty}$;
- validity of the time translation invariance hypothesis by which the material response is assumed to be independent of any temporal shift along the time axis.

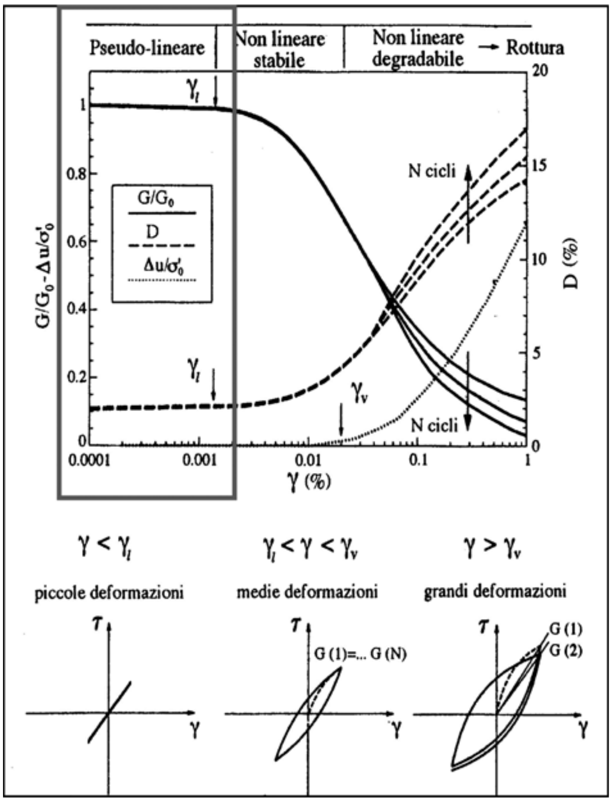

Fig. 1 – Qualitative illustration of mechanical response of geomaterials to a dynamic excitation of cyclic simple shear (*uniaxial* loading). Dependence of the response on the magnitude of the shear strain in relation to the concepts of linear ε^t_l and volumetric ε^t_v cyclic threshold shear strain [from SILVESTRI, 2005].

Fig. 1 – Illustrazione qualitativa della risposta meccanica di geomateriali ad una eccitazione dinamica di taglio semplice (sollecitazione monoassiale). Dipendenza della risposta dall'ampiezza della deformazione di taglio in relazione ai concetti di deformazione lineare ε^t_l e volumetrica ε^t_v ciclica di soglia a taglio [da SILVESTRI, 2005].

Under these assumptions the *Riesz* representation theorem of functional analysis guarantees the existence of a unique relationship between the Cauchy stress tensor $\sigma_{ij}(t)$ and the strain history $\{\varepsilon_{kl}(\tau)\}^t_{-\infty}$ via the following linear functional [CHRISTENSEN, 1971]:

$$\sigma_{ij}(t)=\int_{-\infty}^{t} G_{ijkl}(t-\tau)\frac{d\varepsilon_{kl}(\tau)}{d\tau}d\tau \qquad (1)$$

where the summation convention is implied for repeated indices, ε_{kl} is the infinitesimal strain tensor and $G_{ijkl}(t)$ is the fourth order *relaxation tensor function* of the material. In deriving Eq.(1), also called *Boltzmann's equation*, a continuous strain history has been assumed, however discontinuous strain histories may be handled as well if the integral appearing in Eq.(1) is interpreted in the *Stieltjes* sense.

The relaxation tensor function $G_{ijkl}(t)$ has 81 components; however, only 21 are independent due to the symmetry of the stress and strain tensors in a general anisotropic material. Equation (1) can be "inverted" to obtain the strain tensor ε_{ij} as a function of the stress history as follows:

$$\varepsilon_{ij}(t)=\int_{-\infty}^{t} J_{ijkl}(t-\tau)\frac{d\sigma_{kl}(\tau)}{d\tau}d\tau \qquad (2)$$

where $J_{ijkl}(t)$ is the fourth order *creep tensor function* of the material. For an isotropic, linear, viscoelastic material the creep and relaxation tensor functions have only *two* independent components. In this case constitutive Equation (1) can be rewritten as:

$$\begin{cases} s_{ij}(t)=\int_{-\infty}^{t} 2G_s(t-\tau)\frac{de_{ij}(\tau)}{d\tau}d\tau \\ \sigma_{kk}(t)=\int_{-\infty}^{t} 3G_B(t-\tau)\frac{d\varepsilon_{kk}(\tau)}{d\tau}d\tau \end{cases} \qquad (3)$$

where $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ and $e_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_{kk}/3$ are the components of the deviatoric stress and strain tensors, respectively, and δ_{ij} is the Kronecker symbol. The scalar functions $G_S(t)$ and $G_B(t)$ are the *shear* and *bulk relaxation functions*, respectively. From Eq. (2) it is possible to obtain a pair of relationships analogous to Eq. (3) with the relaxation functions $G_S(t)$ and $G_B(t)$ replaced by the *creep functions* $J_S(t)$ and $J_B(t)$.

The relaxation function $G_S(t)$ represents the shear stress response of a viscoelastic material subjected to a shear strain history specified as a *Heaviside function*, whereas the creep function $J_S(t)$ is the shear strain response of a material subjected to a *Heaviside* shear stress history. Analogous interpretations hold for the bulk relaxation and creep functions $G_B(t)$ and $J_B(t)$. For a *solid* material, the relaxation functions $G_S(t)$ and $G_B(t)$ are assumed to satisfy the following constraints [CHRISTENSEN, 1971]:

1. The amplitude of the relaxation function $G_\alpha(t)$ (with α=S, B) decays exponentially and asymptotically with time to reach a non-zero constant value as $t \to \infty$, that is:

$$G_\alpha(t)=\mathring{G}_\alpha+\hat{G}_\alpha(t) \qquad (4)$$

with $\hat{G}_\alpha(t)\to 0$ and $G_\alpha(t)\to\mathring{G}_\alpha$ as $t\to\infty$. The term $\mathring{G}_\alpha\geq 0$ is known as the *equilibrium response* and represents the elastic component of the relaxation function.

2. The magnitude of the slope of the relaxation function $G_\alpha(t)$ is a continuously decreasing function of time, namely:

$$\left|dG_\alpha(t)/dt\right|_{t=t_1} \leq \left|dG_\alpha(t)/dt\right|_{t=t_2} \text{ for } t_1 > t_2 > 0 \qquad (5)$$

Equation (5) represents a sufficient condition for satisfying the so-called "*fading memory*" hypothesis, which postulates that in a viscoelastic body the

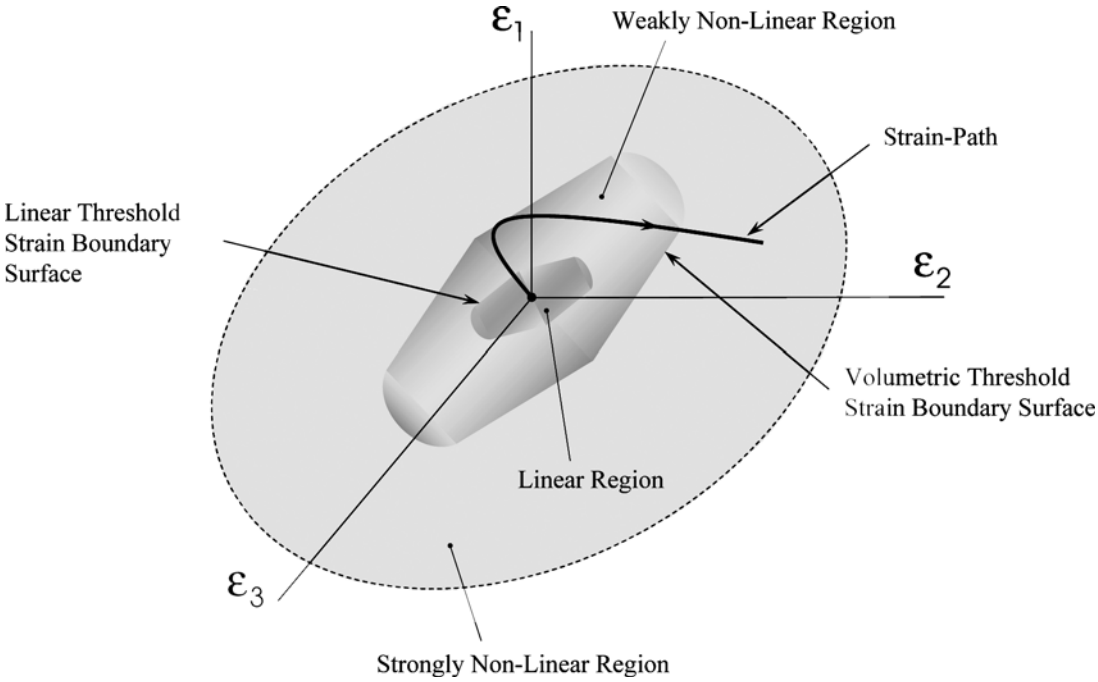

Fig. 2 – Conceptual illustration of mechanical response of geomaterials to a dynamic excitation in *multiaxial* loading. Dependence of response on the *norm* of the deviatoric strain tensor shown through the concepts of linear and volumetric cyclic threshold strain boundary surfaces in the principal strain space [from LAI, 2005].
Fig. 2 – Illustrazione concettuale della risposta meccanica di geomateriali ad una eccitazione dinamica in sollecitazione multi-assiale). Dipendenza della risposta dalla norma del tensore deviatorico delle deformazioni mostrata attraverso i concetti di superfici limiti di deformazione lineare e volumetrica ciclica di soglia nello spazio delle deformazioni principali [da LAI, 2005].

current state of stress depends more strongly on the recent rather than on the remote strain history. Experimental data show that a great number of solids including geomaterials obey the fading memory assumption.

If the strain or stress history is specified as a harmonic function of time, the viscoelastic constitutive relationships assume a very simple form. Let us assume that the strain histories in Eq.(3) are specified by the functions $e_{ij}(t) = e_{0ij} \cdot e^{i\omega t}$ and $\varepsilon_{kk}(t) = \varepsilon_{0kk} \cdot e^{i\omega t}$ where e_{0ij} and ε_{0kk} are the amplitudes of the deviatoric and volumetric strains respectively. Then, these two integral equations degenerate into the following set of algebraic equations:

$$\begin{cases} s_{ij}(t) = 2G_S^*(\omega)\cdot e_{0ij}\cdot e^{i\omega t} \\ \sigma_{kk}(t) = 3G_B^*(\omega)\cdot \varepsilon_{0kk}\cdot e^{i\omega t} \end{cases} \quad (6)$$

where $G^*_S(\omega)$ and $G^*_B(\omega)$ are the *complex shear* and *bulk moduli* respectively. The real part of the complex modulus, $G_{\alpha 1}(\omega)$, is a measure of the energy that the material can store under harmonic oscillations, whereas the imaginary part, $G_{\alpha 2}(\omega)$ is proportional to its capacity for energy dissipation. Again the subscript α=S, B.

The material damping ratio $D_\alpha(\omega)$ provides a normalized measure of the amount of energy dissipated during a cycle of harmonic oscillation for geomaterials. The relationship of $D_\alpha(\omega)$ with the real and imaginary components of the complex modulus is as follows [TSCHOEGL, 1989; LAI and RIX, 2002]:

$$D_\alpha(\omega) = \frac{\Delta W_\alpha^{dissip}(\omega)}{8\pi \cdot W_\alpha^{ave}(\omega)} = \frac{G_{\alpha 2}(\omega)}{2G_{\alpha 1}(\omega)} \quad (7)$$

where $\Delta W_\alpha^{dissip}(\omega)$ is equal to the amount of entropy (per unit volume) produced in one cycle of harmonic loading and due to unrecoverable mechanical work, and $W_\alpha^{ave}(\omega)$ is the average stored energy per cycle of harmonic oscillation.

Combining Eqs. (3), (4), and (6), the real and imaginary components of the complex modulus may be related to the Fourier sine and cosine transforms of the relaxation function through the following relations:

$$G_{\alpha 1}(\omega) = \mathring{G}_\alpha + \omega \int_0^\infty \hat{G}_\alpha(\tau)\sin(\omega\tau)\,d\tau \quad (8)$$

$$G_{\alpha 2}(\omega) = \omega \int_0^\infty \hat{G}_\alpha(\tau)\cos(\omega\tau)\,d\tau \quad (9)$$

Equations (8) and (9) show that the real and the imaginary parts of the complex modulus are not independent. This is expected since the constitutive relation of a linear, isotropic viscoelastic solid is completely defined by only two material functions. In the time domain they may be the relaxation functions $G_S(t)$ and $G_B(t)$. This point concerning the relationship between the real and imaginary parts of the complex modulus is important and will be discussed thoroughly later in this paper.

From Eq.(4) the limiting values of the real and imaginary parts of the complex moduli for $\omega=0$ and $\omega\rightarrow\infty$ can be immediately obtained: [CHRISTENSEN, 1971]

$$G_{\alpha 1}(0)=\mathring{G}_{\alpha} \text{ and } G_{\alpha 2}(0)=0 \text{ for } \omega=0 \quad (10)$$

$$G_{\alpha 1}(\infty)=\mathring{G}_{\alpha}+\hat{G}_{\alpha}(0) \text{ and } G_{\alpha 2}(\infty)=0 \text{ for } \omega\rightarrow\infty \quad (11)$$

The results above show that at *low* and *large* frequencies of excitation, the material behaves as an *elastic solid* with "stiffness" given by the terms $G_{\alpha 1}(0)$ and $G_{\alpha 1}(\infty)$ respectively. Hence, these limiting values of the complex moduli must be non-zero, positive definite constants. Furthermore, since $G_{\alpha 1}(\omega)$ is the storage modulus, for the constitutive model to be physically admissible, $G_{\alpha 1}(\omega)$ must be non-zero, positive for $0<\omega<\infty$. Finally, by means of Eq. (7) and Eqs. (10)-(11), it is possible to obtain the limiting values for the material damping ratio function, which are found to be [MEZA-FAJARDO, 2005]:

$$D_{\alpha}(0)=0 \text{ for } \omega=0 \text{ and } D_{\alpha}(\infty)=0 \text{ for } \omega\rightarrow\infty \quad (12)$$

Equation (12) shows that the spectrum of material damping ratio is defined over a *compact support*, in other words it is defined within constrained limits over a closed and bounded set of frequencies. Damping ratio spectrum enjoys other important mathematical properties which can be appreciated by noticing that Eqs.(8) and (9) relate, through the Fourier transform, the complex modulus to the real-valued relaxation function. It follows that the complex modulus must be a *Hermitian* function and thus the damping ratio spectrum must be an *odd* function. These results play an important role in the solution of the Kramers-Kronig equations as will be shown in the following sections. Besides, they can be very helpful in investigating the frequency-dependence of material damping ratio from experimental measurements.

Propagation of Viscoelastic Waves

As shown in the previous section, in the frequency domain the constitutive relations of linear viscoelasticity become simple and compact algebraic equations, which resemble those of linear elasticity. This resemblance goes even farther as it can be shown that the Fourier-transformed field equations of linear viscoelasticity are formally identical to those associated with linear elasticity except that the elastic shear and bulk moduli G and B are replaced by the complex moduli $G^*_S(\omega)$ and $G^*_B(\omega)$ [CHRISTENSEN, 1971; PIPKIN, 1986].

This analogy between the field equations of linear elasticity and viscoelasticity forms the basis of the *elastic-viscoelastic correspondence principle* [READ, 1950; FUNG, 1965; CHRISTENSEN, 1971; BEN-MENAHEM and SINGH, 2000]. This principle states that elastic solutions to steady state boundary-value problems can be converted into viscoelastic solutions for identical boundary conditions by replacing the elastic shear and bulk moduli G and B with the corresponding complex moduli $G^*_S(\omega)$ and $G^*_B(\omega)$. The validity of the correspondence principle is restricted to problems where the prescribed boundary conditions are *time-invariant*.

Application of this principle to *Navier's* equations of motion for a homogeneous, isotropic elastic medium in absence of body forces, followed by the use of *Helmholtz's* decomposition theorem, yields a pair of wave equations governing the propagation of P and S viscoelastic waves. Their speeds of propagation are complex-valued and are related to constitutive parameters $G^*_S(\omega)$ and $G^*_B(\omega)$ of the medium through the following equations [BEN-MENAHEM and SINGH, 2000]:

$$\begin{cases} V_P^*(\omega)=\sqrt{\dfrac{G_B^*(\omega)+\frac{4}{3}\cdot G_S^*}{\rho}} \\ V_S^*(\omega)=\sqrt{\dfrac{G_S^*(\omega)}{\rho}} \end{cases} \quad (13)$$

These relations simultaneously define *phase velocity* and *attenuation* of monochromatic, P and S bulk viscoelastic waves, as can be shown from the following equations [LAI and RIX, 2002]:

$$\begin{cases} V_\chi(\omega)=\dfrac{[\mathrm{Re}(V_\chi^*)]^2+[\mathrm{Im}(V_\chi^*)]^2}{[\mathrm{Re}(V_\chi^*)]} \\ D_\chi(\omega)=\dfrac{[\mathrm{Re}(V_\chi^*)]\cdot[\mathrm{Im}(V_\chi^*)]}{[\mathrm{Re}(V_\chi^*)]^2-[\mathrm{Im}(V_\chi^*)]^2} \end{cases} \quad (14)$$

where χ = P, S is a subscript denoting longitudinal and transverse wave motion respectively, and $\mathrm{Re}(V^*_\chi)$ and $\mathrm{Im}(V^*_\chi)$ symbolize the real and imaginary part of the complex-valued phase velocities defined by Eq. (13).

Examination of Eq.(14) suggests two important remarks. The first is the frequency-dependence of phase velocity and material damping ratio inherited by constitutive parameters $G^*_S(\omega)$ and $G^*_B(\omega)$ through Eqs. (13) and (14) confirming that a viscoelastic medium is inherently dispersive. The second remark concerns the fact that this equation clearly shows that phase velocity $V_\chi(\omega)$ and material damping ratio $D_\chi(\omega)$ represent an alternative set of constitutive parameters. However since for each mode of deformation (say shear) only one material function is required (say the relaxation function) to completely specify the material functions of a viscoelastic material, $V_\chi(\omega)$ and $D_\chi(\omega)$ may not be prescribed arbitrarily. Thus some kind of relationship is expected to exist between these two functions, for exactly the same reason that the real and imaginary parts of the complex modulus are related. A thorough investigation into the nature of this relationship, its inherent origin and implications is the subject of the following two sections.

Physical principle of causality: Kramers-Kronig relationships

The principle of causality states that in a physical system the reaction to a perturbation can never precede the '*cause*' of the perturbation and constitutes one fundamental postulate of non-relativistic physics. Applied to wave propagation this principle implies that a disturbance originated at a point in a medium (*the source*), is not allowed to arrive at a different point of the same medium (*the observer*) before the time '*d/c*' has elapsed, where '*d*' is the distance between the source and the observer and '*c*' is the speed of propagation of the disturbance in the medium.

In mathematical terms, a real-valued function of time *f(t)* is defined to be *causal*, if it has zero value for $-\infty < t < 0$. If in addition, this function does not have singularities at the origin, then by means of contour integration in the complex plane (see Fig. 3) it can be proved that the real and imaginary parts of its Fourier transform are the Hilbert transform of each other [Tschoegl, 1989; Ben-Menahem and Singh, 2000]. By the same argument it can also be shown that the causality of *f(t)* implies the *analyticity* and *boundness* of its Fourier transform in the lower half of the complex plane for complex-valued frequencies. It is recalled here that a function *f(z)*: **C** ∅ **C** is said to be *analytic* or *holomorphic* in an open set D ⊂ **C** if in D the real and imaginary parts of *f(z)* satisfy the Cauchy-Riemann equations with continuous partial derivatives [Hille, 1973].

Now coming back to our problem, from Eqs.(8) and (9) the real and imaginary parts of the complex modulus are not independent. In fact, by means of a few algebraic manipulations, it is possible to derive the following pair of relationships [Meza-Fajardo, 2005]:

$$G_{\alpha 1}(\omega) - G_{\alpha 1}(\infty) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_{\alpha 2}(\tau)}{\omega - \tau}\,d\tau$$
$$G_{\alpha 2}(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_{\alpha 1}(\tau)}{\omega - \tau}\,d\tau \qquad (15)$$

where again, $G_{\alpha 1}(\omega)$ and $G_{\alpha 2}(\omega)$ are the real and imaginary parts of the complex modulus $G^*_\alpha(\omega)$ respectively and α=S, B. In mathematical terms, Eqs.(15) state that the real and imaginary parts of

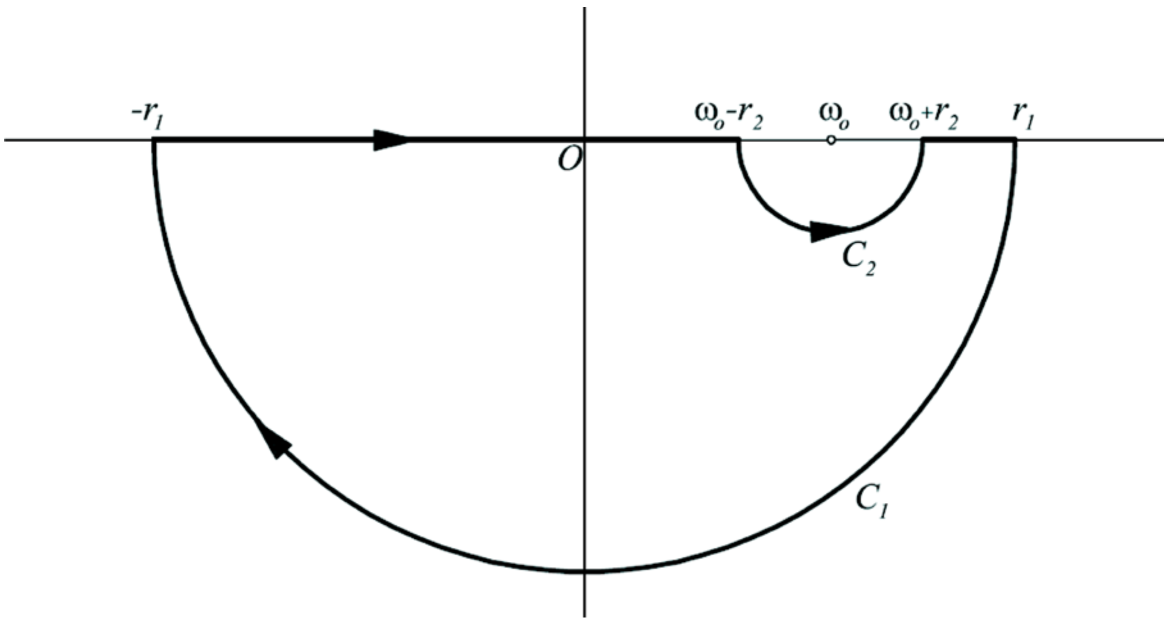

Fig. 3 – Closed contour in the lower part of the complex plane used for the derivation of the Kramers-Kronig relations from the analyticity and boundness of the complex modulus [from Meza-Fajardo, 2005].
Fig. 3 – Circuito chiuso nella parte inferiore del piano complesso utilizzato per la derivazione delle relazioni di Kramers-Kronig a partire dalla analiticità e limitatezza del modulo complesso [da Meza-Fajardo, 2005].

the complex modulus $G^*_\alpha(\omega)$ are the Hilbert transform of each other. Thus, in light of the above statements, the relaxation function $G_\alpha(t)$ is *causal* and the complex modulus $G^*_\alpha(\omega)$ is *analytic* and bounded in the lower half of the complex plane for complex values of ω [BRACEWELL, 1965]. Equations (15) are known in the literature as the *Kramers-Kronig* dispersion relations [TSCHOEGL, 1989; BEN-MENAHEM and SINGH, 2000]. Technically speaking, they are a pair of linear singular integral equations with a Cauchy kernel. The Kramers-Kronig relations are important because they state that in a viscoelastic material the real and imaginary parts of the complex modulus $G^*_\alpha(\omega)$ cannot be specified independently and thus if one of the two components is arbitrarily prescribed, the other is automatically determined by the causality constraint. Furthermore, they constitute the necessary and sufficient conditions for the material functions $G^*_\alpha(\omega)$ and $G_\alpha(t)$ to satisfy the fundamental principle of causality.

In viscoelastodynamics the Kramers-Kronig relationships are more conveniently written in terms of wave propagation parameters $V_\chi(\omega)$ and $D_\chi(\omega)$ using Eqs.(13) and (14). In this case the Hilbert transform pair is constituted by the real and imaginary parts of the complex wavenumber ω/V^*_χ for χ = P, S [AKI and RICHARDS, 2002].

From a practical point of view, solutions of the Kramers-Kronig relationships would be of great interest because they would provide an effective tool to investigate the frequency-dependence of low-strain, constitutive parameters in dissipative materials such as soils and rocks. They would also offer an alternative approach in the experimental determination of the low-strain dynamic properties of geomaterials and of their spectra.

Approximate solutions of Kramers-Kronig relationships

A widely used approximate solution of the Kramers-Kronig relationships is the one adopted in seismology [KJARTANSSON 1979; KENNETT, 1983; KEILIS-BOROK, 1989; BEN-MENAHEM and SINGH, 2000; AKI and RICHARDS, 2002] based on the assumption that material damping ratio is rate-independent (i.e. hysteretic) over the seismic band (~ from 0.001-10 Hz). The dispersion relation that follows from this hypothesis and that satisfies the principle of causality can be written as follows:

$$V_\chi(\omega)=\frac{V_\chi(\omega_{ref})}{1+\frac{2D_\chi}{\pi}\log\left(\frac{\omega_{ref}}{\omega}\right)} \qquad (16)$$

where ω_{ref} denotes an angular reference frequency which is usually set equal to 2π rad/sec.

Equation (16) is applicable only for weakly dissipative media and for the frequency range ~0.001-10 Hz [LIU *et al.*, 1976], which corresponds approximately to the seismic band. In this range of frequencies material damping ratio $D_\chi(\omega)$ is assumed to be a constant and therefore frequency independent.

Figure 4 shows the dispersion effects predicted by Eq.(16). The normalized phase velocity reported in the figure corresponds to the ratio $V_\chi(\omega)/V_\chi(\omega_{ref})$ with $\omega_{ref} = 2\pi$. As expected, this ratio increases with frequency and the effect is more pronounced for higher values of damping ratio. Thus, according to Eq.(16), the influence of material dispersion is stronger at high frequencies and for larger values of (low-strain) energy dissipation.

Several seismological [BERCKHEMER *et al.*, 1982; JONES, 1986; FUKUSHIMA *et al.*, 1992; JACKSON *et al.*, 2002; SATOH, 2006] and geotechnical [ZAVORAL and CAMPANELLA, 1994; LEROUEIL and MARQUES, 1996; D'ONOFRIO *et al.*, 1999; STOKOE *et al.*, 1999; RIX and MENG, 2005] studies have questioned the validity of the hypothesis of hysteretic damping ratio for geomaterials over the seismic band. Certainly the assumption of rate-independency of material damping ratio is not applicable systematically to all types of geomaterials. Even the definition of the frequency range of the seismic band is somehow loose and it is not immediately obvious, in the absence of a rigorous solution of the Kramers-Kronig equations, what type of influence is exerted on the dispersion relation (16), by a shift of the lower and upper bounds of the so-called "*seismic band*".

Alternative, approximate solutions of the Kramers-Kronig relations have been obtained by other researchers. Among these it is worth mentioning the contribution of BOOIJ and THOONE [1982] who assumed that the derivative of the real part of the complex modulus with respect to the logarithm of frequency is constant (or varies slowly with frequency), namely:

$$\frac{dG_{\chi 1}(\omega)}{d\log(\omega)}=constant \qquad (17)$$

which yields the following approximate solutions of the Kramers-Kronig relationships:

$$\begin{cases} D_\chi(\omega)=\frac{\pi}{4}\left(\frac{d\log\left|G^*_\chi(\omega)\right|}{d\log(\omega)}\right) \\ \left|G^*_\chi(\omega_2)\right|=\left|G^*_\chi(\omega_1)\right|\exp\left(\frac{4}{\pi}\int_{\omega_1}^{\omega_2}D_\chi(\omega)d\log(\omega)\right) \end{cases} \qquad (18)$$

where χ=S, P. The second of Eqs.(18) only allows the computation of relative values of the complex shear modulus from the damping ratio spectrum

$D_\chi(\omega)$ and a reference value of $|G^*_\chi(\omega_1)|$ is required to calculate the variation of this parameter with frequency. The validity of Eqs.(18) includes as a special case the situation of rate-independent (i.e. hysteretic) damping ratio. MENG [2003] adopted these approximate solutions of the Kramers-Kronig relationships to model strain-rate effects of remolded kaolin and fine-grained natural soils obtaining satisfactory results.

Exact solutions of Kramers-Kronig relationships

Overview

To the best knowledge of the authors, exact solutions of the Kramers-Kronig relationships expressed in terms of $V_\chi(\omega)$ and $D_\chi(\omega)$have not yet been found. As mentioned above, they form a pair of linear, singular integral equations with Cauchy kernel which constitute a rather difficult mathematical problem to solve. So far only approximate solutions are available whose usefulness is severely limited by the various assumptions considered in their derivation. A significant step forward toward the solution of this problem has recently been accomplished by MEZA-FAJARDO [2005] who devised for it four solution strategies. The first one is based on the transformation of the singular integral Equation (15a) into a *Fredholm* integral equation of the second kind which corresponds to a simpler mathematical problem. This strategy was only partially successful because a closed-form solution of this integral equation could not be found as its integrand does not have any special structure, it is not of a convolution type and the kernel is neither symmetric nor degenerate. Without an explicit representation of the damping ratio spectrum $D_\chi(\omega)$, the method of successive approximations could not be applied either. The only available solution scheme available for this equation is the *Nystrom* method which is based on numerical integration [TRICOMI, 1985]. However a Fredholm integral equation of the second kind is a well-posed mathematical problem and thus its numerical solution with the Nystrom method is stable and the convergence is guaranteed.

The second and third strategy devised by MEZA-FAJARDO [2005] solve exactly and in closed-form, the Kramers-Kronig relationships considering $V_\chi(\omega)$ as the dependent and $D_\chi(\omega)$ as the independent material function. One method solves the problem by means of the theory of singular integral equations [MUSKHELISHVILI, 1992] whereas the other is based on exploiting the mathematical implications of the

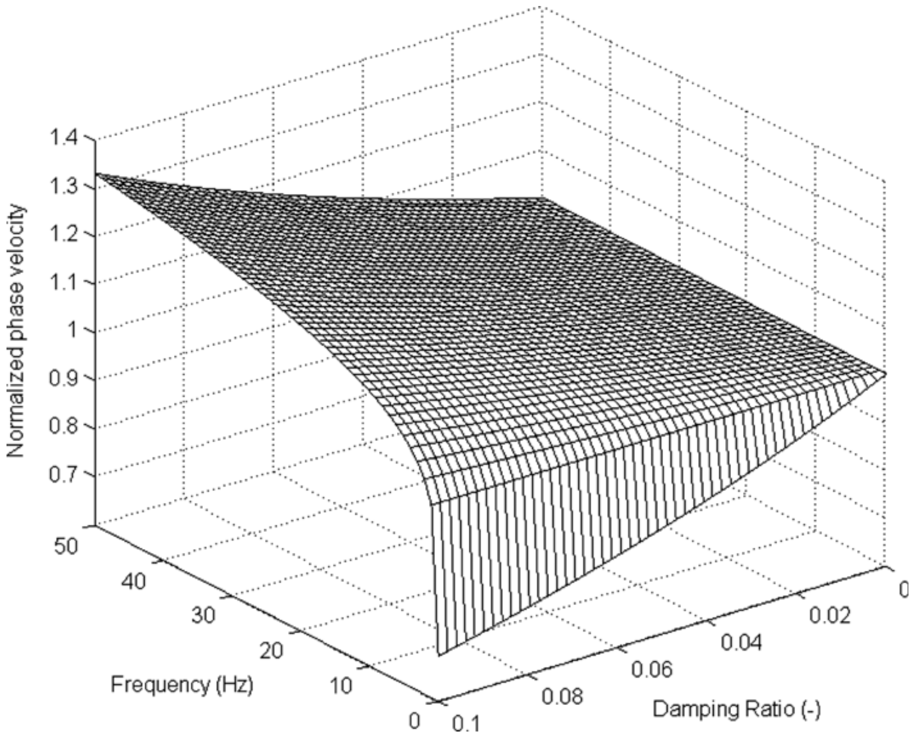

Fig. 4 – Frequency-dependence of phase velocity of viscoelastic bulk waves predicted by the dispersion relation represented by Eq.(16) and based on assuming rate-independent (i.e. hysteretic) material damping ratio over the seismic frequency band.
Fig. 4 – Dipendenza dalla frequenza della velocità di fase di onde viscoelastiche di volume predetta dalla relazione di dispersione rappresentata dall'Eq. (16) e basata sull'assunzione del rapporto di smorzamento indipendente dalla velocità di applicazione del carico (cioè isteretico) nella banda di frequenza sismica.

principle of physical causality. Finally the fourth strategy is based on solving the Kramers-Kronig relations by inverting the role of $V_\chi(\omega)$ and $D_\chi(\omega)$ as dependent and independent material functions. In this case, also, the solution that is obtained is exact and expressed in closed-form.

Details about the solution of the Kramers-Kronig relations will now be illustrated just for two methodologies out of the four introduced. The reader interested in knowing about the other two approaches is refereed to the original document by MEZA-FAJARDO [2005] or alternatively to the paper by MEZA-FAJARDO and LAI [2006].

Solution by exploiting the mathematics of physical causality

As previously stated, the causality of the relaxation function $G_\alpha(t)$ implies the analyticity and boundness of the complex modulus $G^*_\alpha(\omega)$ in the lower half of the complex plane for complex values of the frequencies (α=S,B). This result, extended to $G_\chi(t)$ and $G^*_\chi(\omega)$ with χ=S, P, can be profitably exploited by considering the polar representation of the complex modulus $G^*_\chi(\omega)$:

$$\begin{cases} G^*_\chi(\omega)=\left|G^*_\chi(\omega)\right|\exp\left[i\varphi_\chi(\omega)\right] \\ \varphi_\chi(\omega)=\tan^{-1}\left[G_{\chi^2}(\omega)/G_{\chi^1}(\omega)\right]=\tan^{-1}\left[2D_\chi(\omega)\right] \end{cases} \qquad (19a,b)$$

Taking the natural logarithm of both sides of Eq. (19a) yields:

$$\ln G^*_\chi(\omega)=\ln\left|G^*_\chi(\omega)\right|+i\varphi_\chi(\omega) \qquad (20)$$

Considering the limiting values of the complex modulus given by Eqs. (10) and (11), it is easy to infer that for a viscoelastic solid $G^*_\chi(\omega)$ is non-zero and bounded for all (real) ω. Furthermore this function is continuous. These conclusions can also be deduced on physical grounds [MEZA-FAJARDO, 2005]. It follows by *analytic continuation* [HILLE, 1973] that $G^*_\chi(\omega^*)$ is *holomorphic* and bounded in the lower part of the complex plane including the real axis. These properties carry over to the function $\ln[G^*_\chi(\omega^*)]$ defined by Eq. (20) and thus, by the *principle of causality* discussed in section 4, the real and imaginary parts of $\ln[G^*_\chi(\omega)]$ are also Hilbert Transforms of each other, just as $G_{\alpha1}(\omega)$ and $G_{\alpha2}(\omega)$ are in Eqs.(15), namely:

$$\begin{aligned} &\ln\left|G^*_\chi(\omega)\right|=\ln\left|G^*_\chi(\infty)\right|-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tan^{-1}\left[2D_\chi(\tau)\right]}{\tau-\omega}\mathrm{d}\tau \\ &\tan^{-1}\left[2D_\chi(\omega)\right]=\tan^{-1}\left[2D_\chi(\infty)\right]+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln\left|G^*_\chi(\tau)\right|}{\tau-\omega}\,\mathrm{d}\tau \end{aligned} \qquad (21)$$

Considering $G^*_\chi(\omega)$ decomposed in its real and imaginary parts and substituting Eq.(7) into Eq. (21a) yields:

$$\ln\left[G_{\chi^1}(\omega)\sqrt{1+4D^2_\chi(\omega)}\right]-\ln\left[G_{\chi^1}(\infty)\sqrt{1+4D^2_\chi(\infty)}\right]= =\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tan^{-1}\left[2D_\chi(\tau)\right]}{\omega-\tau}\mathrm{d}\tau \qquad (22)$$

Taking the exponential of both sides of this equation and using Eq.(12) with α=S, B replaced by χ=S, P yields:

$$\frac{G_{\chi^1}(\omega)}{G_{\chi^1}(\infty)}=\frac{1}{\sqrt{1+4D^2_\chi(\omega)}}\,e^{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tan^{-1}\left[2D_\chi(\tau)\right]}{\omega-\tau}\mathrm{d}\tau} \qquad (23)$$

Now considering that $\tan^{-1}[2D_\chi(\tau)]$ is an odd function and that the odd part of $1/(\omega-\tau)$ is $\tau/(\omega^2-\tau^2)$, the solution for $G_{\chi1}(\omega)$ is given by:

$$\frac{G_{\chi^1}(\omega)}{G_{\chi^1}(\infty)}=\frac{1}{\sqrt{1+4D^2_\chi(\omega)}}\,e^{\frac{2}{\pi}\int_{0}^{\infty}\frac{\tau\tan^{-1}\left[2D_\chi(\tau)\right]}{\omega^2-\tau^2}\mathrm{d}\tau} \qquad (24)$$

In terms of the material function $G_{\chi1}(\omega)$ evaluated for $\omega = 0$, the solution becomes [MEZA-FAJARDO, 2005]:

$$\frac{G_{\chi^1}(\omega)}{G_{\chi^1}(0)}=\frac{e^{\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega^2\tan^{-1}\left[2D_\chi(\tau)\right]}{\tau\,(\omega^2-\tau^2)}\mathrm{d}\tau}}{\sqrt{1+4D^2_\chi(\omega)}} \qquad (25)$$

To complete the derivation, it is now convenient to express the above solution of the Kramers-Kronig relation in terms of phase velocity of monochromatic P and S viscoelastic waves. For this purpose Eq.(14a) is more conveniently re-written in the following form which emphasizes the dependence of phase velocity $V_\chi(\omega)$ dispersion on the dissipative properties $D_\chi(\omega)$ of the medium [LAI and RIX, 2002]:

$$V_\chi(\omega)=\sqrt{\frac{G_{\chi^1}(\omega)}{\rho}}\sqrt{\frac{2\sqrt{1+4D^2_\chi(\omega)}}{1+\sqrt{1+4D^2_\chi(\omega)}}} \qquad (26)$$

where ρ is the time-independent mass density of the medium.

Finally, substitution of Eq.(26) into Eq.(25) leads to the desired, exact solution of the Kramers-Kronig relations expressed here in terms of phase velocity of P and S waves as a function of material damping ratio $D_\chi(\omega)$ and with reference to $V_\chi(0)$:

$$\frac{V_\chi(\omega)}{V_\chi(0)}=\sqrt{\frac{2\sqrt{1+4D^2_\chi(\omega)}}{1+\sqrt{1+4D^2_\chi(\omega)}}\,e^{\frac{2}{\pi}\int_{0}^{\infty}\frac{\omega^2\tan^{-1}\left[2D_\chi(\tau)\right]}{\tau\,(\omega^2-\tau^2)}\mathrm{d}\tau}} \qquad (27)$$

The relationship between the limiting values of phase velocity for $\omega = 0$ and $\omega\varnothing\infty$, namely $V_\chi(0)$ and $V_\chi(\infty)$, is given by the following expression [Meza-Fajardo and Lai, 2006]:

$$\frac{V_\chi(\infty)}{V_\chi(0)} = \frac{G_{\chi^1}(\infty)}{G_{\chi^1}(0)} = \exp\left(\frac{2}{\pi}\int_0^\infty \frac{\tan^{-1}[2D_\chi(\tau)]}{\tau}\mathrm{d}\tau\right) \quad (28)$$

This latter result indicates that $V_\chi(\infty) \geq V_\chi(0)$ if $D_\chi(\omega) \geq 0$ for all ω. In fact, it can be shown that in general for the frequency bands where damping ratio is greater than zero, the phase velocity must be an increasing function of frequency. Furthermore, if $V_\chi(\infty) = V_\chi(0)$ then $D_\chi(\omega) = 0$ for all ω which corresponds to the case of a perfectly elastic material [Meza-Fajardo, 2005].

An inverse solution: damping ratio spectrum as a function of phase velocity dispersion

In the previous section, the Kramers-Kronig integral equations were explicitly solved to obtain frequency-dependent phase velocity $V_\chi(\omega)$ of P and S waves as a function of material damping ratio function $D_\chi(\omega)$. Now the derivation of a counterpart solution of the Kramers-Kronig equations will be shown, namely a relation where the function $D_\chi(\omega)$ is expressed as a unique function of the frequency-dependent phase velocity $V_\chi(\omega)$. As a matter of fact, this derivation turns out to be less involved than the one where $D_\chi(\omega)$ is assumed to be the independent material function.

The starting point for obtaining the solution is the Kramers-Kronig equations written in terms of wave propagation parameters $V_\chi(\omega)$ and $\alpha_\chi(\omega)$ where the latter is the frequency-dependent a*ttenuation coefficient* associated with the χ-wave [Aki and Richards, 2002]:

$$\alpha_\chi(\omega) - \alpha_\chi(0) = \frac{\omega}{\pi}\int_{\infty}^{+\infty}\left(\frac{1}{V_\chi(\tau)} - \frac{1}{V_\chi(\infty)}\right)\frac{\mathrm{d}\tau}{\omega-\tau}$$
$$\omega\left(\frac{1}{V_\chi(\omega)} - \frac{1}{V_\chi(\infty)}\right) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha_\chi(\tau)\mathrm{d}\tau}{\omega-\tau} \quad (29)$$

Equation (29a) can be simplified by noticing that the Hilbert transform of a constant is zero. In the previous section it was stated that for a viscoelastic solid $G^*_\chi(0) \neq 0$ from which it can be easily shown that $\alpha_\chi(0) = 0$ [Meza-Fajardo, 2005]. Thus from Eq. (29) it follows:

$$\alpha_\chi(\omega) = \frac{\omega}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{V_\chi(\tau)}\right)\frac{\mathrm{d}\tau}{\omega-\tau} \quad (30)$$

On the other hand, the attenuation function $\alpha_\chi(\omega)$ can be expressed as a function of phase velocity $V_\chi(\omega)$ and damping ratio $D_\chi(\omega)$ via the following relation [Lai and Rix, 2002]:

$$\alpha_\chi(\omega) = \frac{\omega}{V_\chi(\omega)}\frac{\sqrt{1+4D_\chi^2(\omega)}-1}{2D_\chi(\omega)} \quad (31)$$

By equating Eqs. (30) and (31) the following result is obtained:

$$2D_\chi(\omega)M(\omega) = 1 - \sqrt{1+4D_\chi^2(\omega)} \quad (32)$$

where the auxiliary function $M(\omega)$ is defined as follows:

$$M(\omega) = \frac{V_\chi(\omega)}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{V_\chi(\tau)}\right]\frac{\mathrm{d}\tau}{\tau-\omega} \quad (33)$$

Solving the quadratic Equation (32) for $D_\chi(\omega)$ leads to the following result:

$$D_\chi(\omega) = \frac{M(\omega)}{M^2(\omega)-1} \quad (34)$$

Then, the explicit relation of frequency-dependent damping ratio $D_\chi(\omega)$ as a function of phase velocity dispersion is given by:

$$D_\chi(\omega) = \frac{\frac{V_\chi(\omega)}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{V_\chi(\tau)}\right)\frac{\mathrm{d}\tau}{\tau-\omega}}{\left[\frac{V_\chi(\omega)}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{V_\chi(\tau)}\right)\frac{\mathrm{d}\tau}{\tau-\omega}\right]^2 - 1} \quad (35)$$

Finally, this expression can be further simplified by noticing from Eq. (26) that phase velocity function $V_\chi(\omega)$ is an *even* function of frequency and that the even part of $1/(\tau\text{-}\omega)$ is $\omega/(\tau^2\text{-}\omega^2)$:

$$D_\chi(\omega) = \frac{\frac{2\omega V_\chi(\omega)}{\pi}\int_0^{\infty}\left(\frac{1}{V_\chi(\tau)}\right)\frac{\mathrm{d}\tau}{\tau^2-\omega^2}}{\left[\frac{2\omega V_\chi(\omega)}{\pi}\int_0^{\infty}\left(\frac{1}{V_\chi(\tau)}\right)\frac{\mathrm{d}\tau}{\tau^2-\omega^2}\right]^2 - 1} \quad (36)$$

Equation (36) is instructive as it shows that the frequency-dependent damping ratio $D_\chi(\omega)$ can entirely be determined from measurements of phase velocity dispersion. Although Eqs.(27) and (36) show that $V_\chi(\omega)$ and $D_\chi(\omega)$ are mathematically equivalent in defining the material response of a viscoelastic solid, from an experimental point of view the determination of damping ratio spectrum $D_\chi(\omega)$ is less straightforward and affected by higher uncertainties than the measurement of $V_\chi(\omega)$. Therefore Eq.(36) represents an effective and inexpensive means of computing damping-ratio frequency laws exclusively from phase velocity measurements.

It is also noted that whereas in Eq.(27) the dispersion law $V_\chi(\omega)$ of phase velocity is defined with respect to either $V_\chi(0)$ or $V_\chi(\infty)$, damping ratio spectrum $D_\chi(\omega)$ calculated from Eq.(36) does not require knowledge of the limiting values of $D_\chi(\omega)$. Eq. (12) explains the reason for this result, which lies in the predefined limiting values of $D_\chi(\omega)$.

Applications and validation of the exact solutions

The results obtained are here compared with well-known approximate solutions of the Kramers-Kronig relationships as well as with some experimental data that have been recently published in the literature. For all the cases presented in this section, damping ratio and phase velocity functions $D_\chi(\omega)$ and $V_\chi(\omega)$ correspond to shear wave motion, therefore the subscript will be dropped in the mathematical notation.

Hysteretic (i.e. rate-independent) damping ratio

A natural verification of the rigorous solutions of the Kramers-Kronig relationships is that associated with the dispersion relation widely used in seismology and based on the assumption that over the seismic band damping ratio is a rate-independent, hysteretic function. It should be pointed out however that the Kramers-Kronig equations admit no solution for a damping ratio that is constant for the whole frequency spectrum [AKI and RICHARDS, 2002] and therefore some type of frequency-dependence should be admitted outside the seismic band. As a matter of fact it was shown by Eq. (12) that for a viscoelastic solid material damping ratio should vanish at zero and at very high frequencies.

Thus, in order to account for these constraints, a damping ratio function $D(\omega)$ is constructed such that it is constant over the seismic band specified in the range 0.001–10 Hz and then it goes to zero smoothly with sinusoidal ascending and descending branches. Figure 5 shows the plot of such a function. For the constant, non-zero value of $D(\omega)$ a value of 0.5% was assumed, which is typical for a rock (at low-strain levels) and allows a comparison with the results presented by LIU *et al.*, [1976] and reported also by AKI and RICHARDS [2002]. Figure 6 shows the phase velocity dispersion curve calculated with both the exact solution given by Eq. (27) and the simplified solution given by Eq. (16). In the calculations, the phase velocity at the reference frequency $\omega_{ref} = 2\pi$ was considered to have a value of 4.5km/s. The results show an excellent agreement between the two solutions. They both predict the same monotonically increasing function $V(\omega)$ over the seismic frequency band where material damping ratio is a constant. Based on this outcome, Eq.(27) could be profitably used to investigate the influence of the lower and upper bounds of the seismic band on the dispersion curve defining the frequency dependence.

In recent years several studies have investigated the influence of strain-rate effects on *fine-grained*

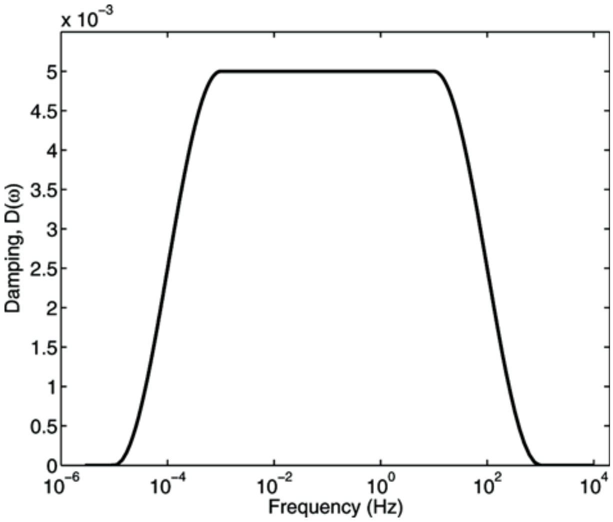

Fig. 5 – Theoretical model constructed for frequency-independent (i.e. hysteretic) damping ratio over the seismic band of 0.001–10 Hz.
Fig. 5 – Modello teorico costruito per un rapporto di smorzamento indipendente dalla frequenza (cioè isteretico) nella banda sismica tra 0.001–10 Hz.

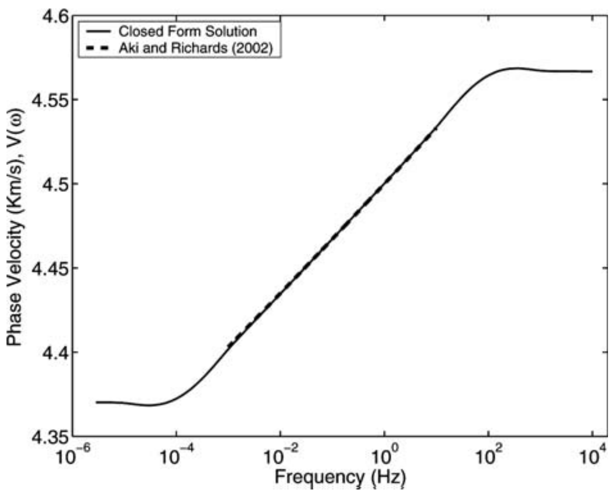

Fig. 6 – Comparison of phase velocity dispersion curves obtained for rate-independent (i.e. hysteretic) damping ratio using exact and approximate solutions. The exact solution has been calculated via Eq.(27) adopting the damping ratio function shown in Fig. 5. The approximate solution has been determined using Eq.(16).
Fig. 6 – Confronto tra curve di dispersione della velocità di fase ottenute per un rapporto di smorzamento indipendente dalla velocità di applicazione del carico (cioè isteretico) utilizzando la soluzione esatta e approssimata. La soluzione esatta è stata calcolata mediante l' Eq.(27) adottando la funzione rapporto di smorzamento mostrata in Fig. 5. La soluzione approssimata è stata determinata utilizzando l'Eq.(16).

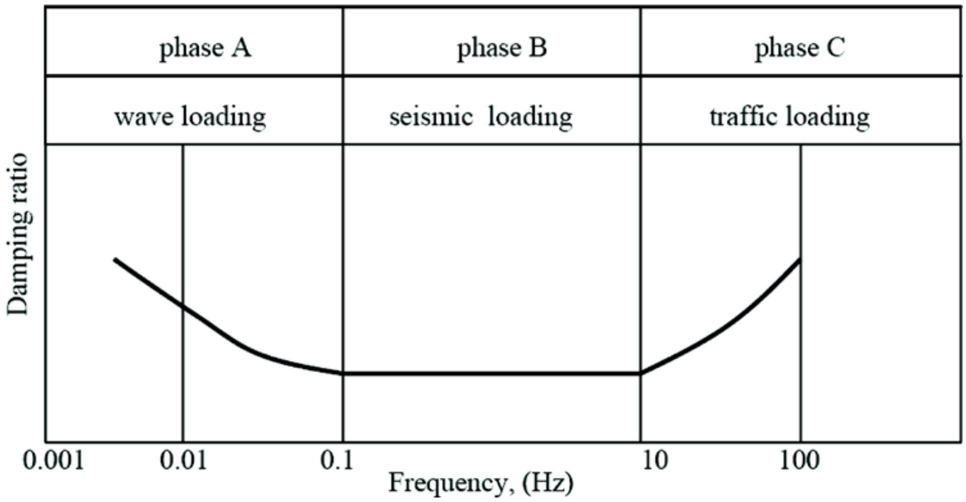

Fig. 7 – Qualitative description of the frequency-dependence of small-strain, damping ratio in fine-grained soils [after SHIBUYA *et al.*, 1995].
Fig. 7 – Descrizione qualitativa della dipendenza dalla frequenza del rapporto di smorzamento a piccole deformazioni in terreni a grana fine [da SHIBUYA et al.*, 1995].*

soils over a wide range of frequencies and a large amount of experimental data is now available [KIM, 1991; SHIBUYA *et al.*, 1995; ISHIHARA, 1996; D'ONOFRIO *et al.*, 1999; STOKOE *et al.*, 1999; LEROUEIL and MARQUES 1996; MENG, 2003]. In particular SHIBUYA *et al.* [1995], referring to experimental laboratory data obtained in cyclic loading tests, proposed for the small-strain damping ratio a frequency-dependence law described qualitatively by the diagram shown in Fig. 7. In this chart the frequency spectrum is divided into three zones or phases depending on the type of dissipation mechanism controlling the energy losses at a particular frequency band.

According to SHIBUYA *et al.* [1995], at very low frequencies (i.e. phase A) such as those associated with wave loading, soil dissipation occurs through creep and tends to increase as the frequency decreases. Phase B corresponds to the seismic frequency band and energy losses are rate-independent or hysteretic in character. At higher frequencies (i.e. phase C) damping ratio increases with the strain rate and this is attributed to an enhanced contribution of pore fluid viscosity to the overall energy losses. The conceptual diagram of SHIBUYA *et al.* [1995] has recently been validated by the damping ratio frequency relations that MENG [2003] obtained for clayey sands and for sandy fat clays by using resonant and non-resonant column tests.

The damping ratio spectrum of Figure 7 was extended outside its domain of definition to cover the whole frequency range. To this purpose smooth ascending and descending branches have been added to the outer domains of the graph so as to make damping ratio vanish at zero and very high frequency as shown in Fig. 8. The phase velocity dispersion curve calculated from Eq. (27) using the

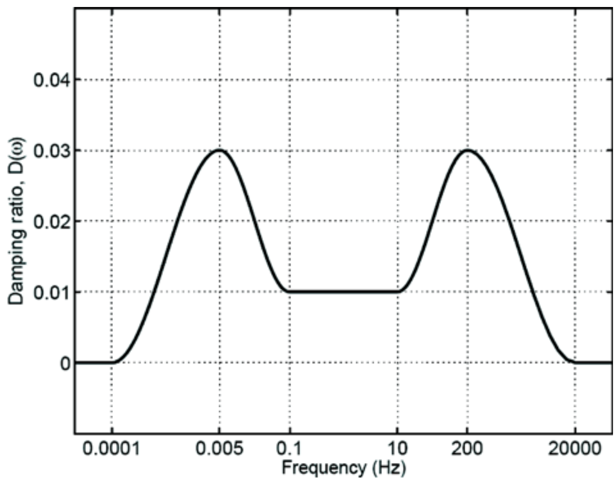

Fig. 8 – Extension outside the domain of definition of the small-strain, damping ratio spectrum proposed in qualitative terms by SHIBUYA *et al.* [1995] for fine-grained soils.
Fig. 8 – Estensione all'esterno del dominio di definizione dello spettro del rapporto di smorzamento a piccole deformazioni proposto in termini qualitativi da SHIBUYA et al. *[1995] per terreni a grana fine.*

damping ratio function $D(\omega)$ of Fig. 8 is illustrated in Fig. 9. It is interesting to observe that the increase of phase velocity with frequency is more pronounced in regions of the spectrum where the damping ratio function is not a constant but varies. The explanation for this behaviour is straightforward in light of Eq. (27). In fact from this equation phase velocity $V(\omega)$ is obtained by direct integration of the damping ratio function $D(\omega)$ and the ascending and descending branches of $D(\omega)$ along phases A and C (see Fig. 7) lead to greater integrated areas under the damping ratio function and thus to higher changes in phase velocity. This implies that

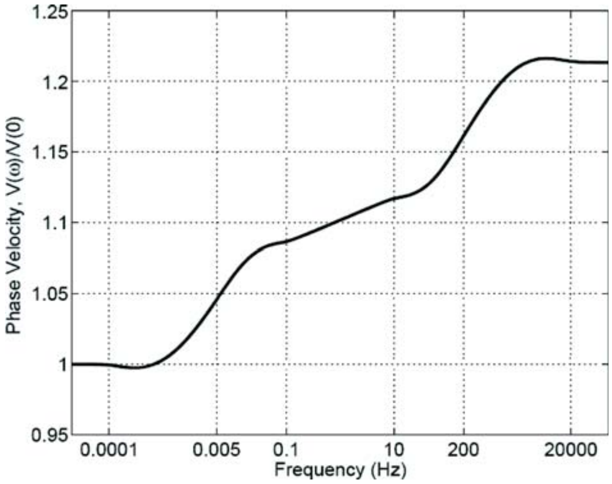

Fig. 9 – Phase velocity dispersion curve obtained from Eq. (27) using the damping ratio function $D(\omega)$ shown in Fig. 8.

Fig. 9 – Curva di dispersione della velocità di fase ottenuta a partire dall'Eq. (27) utilizzando la funzione rapporto di smorzamento D(ω) mostrata in Fig. 8.

the frequency-dependence of phase velocity is stronger if a material dissipates more energy.

Experimental results

The exact solutions of the Kramers-Kronig equations illustrated have been applied to reproduce the experimental results that RIX and MENG [2005] have recently obtained while attempting to determine the frequency-dependence laws of complex shear modulus amplitude $|G^*(\omega)|$ and shear damping ratio $D(\omega)$ of remoulded kaolin and natural sandy silty clay over a continuous frequency range from 0.01 to 30 Hz by using a non-resonance testing method.

In order to apply the theoretical solutions a polynomial was first fitted to the experimental data using the standard least squares method. Extensions for values of $|G^*(\omega)|$ and $D(\omega)$ outside the range of experimental frequencies (i.e. from 0.01 to 30 Hz) were added using regular pieces of smooth functions. For damping ratio these were constituted by two branches of decaying sinusoids with $D(\omega)$ equal to zero for $\omega = 0$ and $\omega \rightarrow \infty$, just about the same way it was done in the previous section. At the boundary values of about 0.01 and 30 Hz the first derivative of $D(\omega)$ was required to vanish assuming local maxima are attained at these points. The prolongation for $|G^*(\omega)|$ required to compute material damping ratio from Eq.(21b) was constructed by considering that this should be an increasing function of frequency to reach constant, rate-independent values as ω approaches zero and infinity [MEZA-FAJARDO and LAI, 2006].

The composite, fitted curves of the experimental data that RIX and MENG [2005] have determined for $|G^*(\omega)|$ and $D(\omega)$ were then employed to obtain the results shown in Figs. 10 and 11, which were calculated by alternatively using Eqs. (21b) and (23). In particular the smooth curve shown in Fig. 10a was determined by applying Eq.(21b) to the experimental data of Fig. 10b which were initially fitted by a prolonged polynomial using the least square method. Similarly, the curve shown in Fig. 10b was obtained by applying Eq. (23) to the experimental damping ratio data of Fig. 10a after they had been fitted by an appropriate polynomial curve. A similar procedure was used to produce Figure 11(a) and (b).

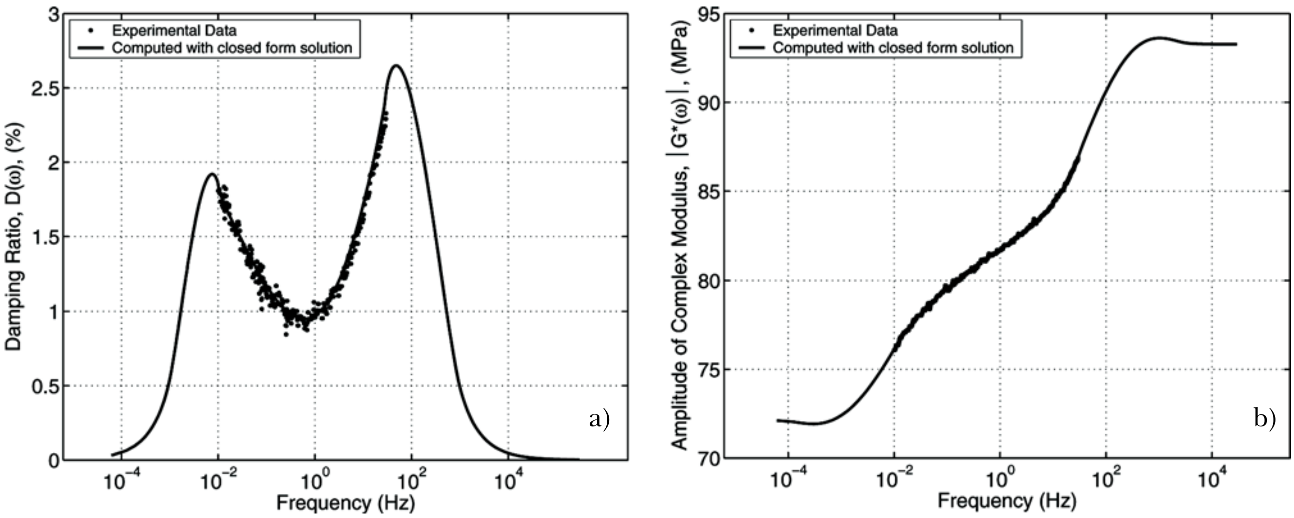

Fig. 10 – Comparison between experimental data obtained by RIX and MENG [2005] in applying the non-resonant column testing method to remoulded kaolin and theoretical predictions from Eqs. (21b) and (23) [from MEZA-FAJARDO and LAI, 2006].

Fig. 10 – Confronto tra i dati sperimentali ottenuti da RIX E MENG [2005] nell'applicazione del metodo di prova con la colonna non risonante al caolino rimaneggiato e le previsioni teoriche ottenute a partire dalle Eqq. (21b) and (23) [da MEZA-FAJARDO and LAI, 2006].

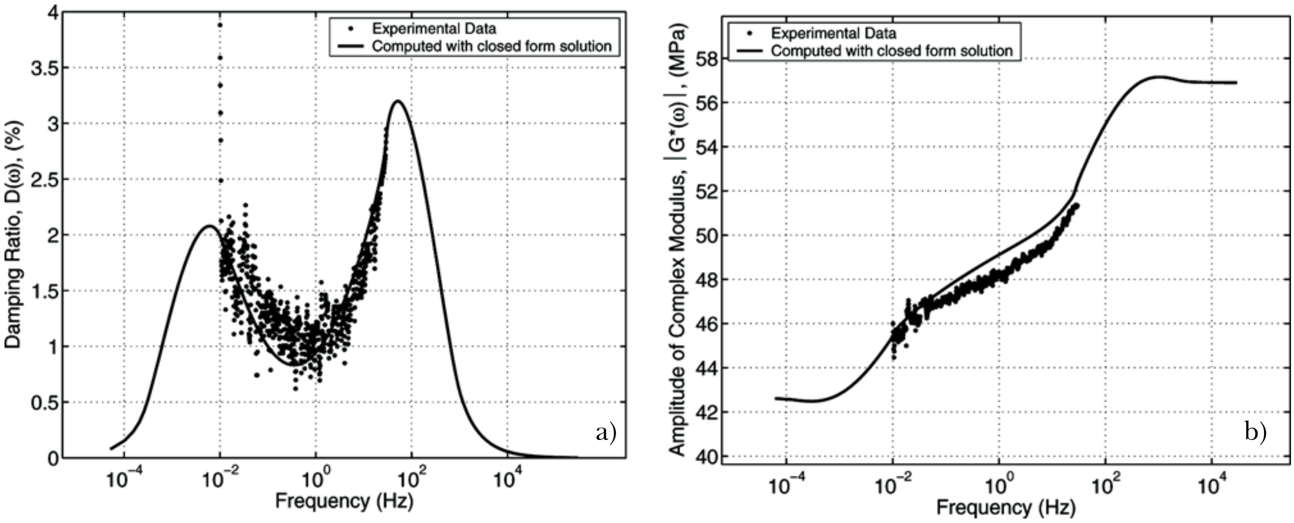

Fig. 11 – Comparison between experimental data obtained by RIX and MENG [2005] in applying the non-resonant testing method to natural sandy silty clay and theoretical predictions from Eqs. (21b) and (23) [from MEZA-FAJARDO and LAI, 2006].
Fig. 11 – Confronto tra i dati sperimentali ottenuti da RIX e MENG [2005] nell'applicazione del metodo di prova con la colonna non risonante ad una argilla limoso-sabbiosa naturale e le previsioni teoriche ottenute a partire dalle Eqq. (21b) and (23) [da MEZA-FAJARDO and LAI, 2006].

The results show an excellent agreement between the experimental data and the curves obtained from the theoretical predictions. Figures 10 and 11 show that unlike coarse-grained geomaterials, the small strain dynamic response of fine-grained soils is significantly affected by the rate of loading. Finally, it is important to remark here that the data measured by RIX and MENG [2005] were obtained by exploiting the elastic-viscoelastic corresponding principle and thus they are compatible with physical causality as confirmed by the excellent agreement with the closed-form solutions of the Kramers-Kronig equations.

Concluding remarks

In this paper, after reviewing the derivation of Kramers-Kronig integral equations, which represent the necessary and sufficient conditions for a mechanical disturbance propagating in a linear viscoelastic medium to satisfy the fundamental principle of physical causality, both approximate and recently obtained exact solutions of these equations have been presented. The exact solutions of Kramers-Kronig equations allow frequency-dependent phase velocity to be computed from damping ratio spectrum and inversely frequency-dependent damping ratio from phase velocity dispersion.

They have been derived with no simplifying assumptions other than those of standard viscoelasticity theory complemented by the fading memory hypothesis. It was then shown that they are compatible with approximate solutions of Kramers-Kronig relations and in particular with that based on the hypothesis of rate-independent (i.e. hysteretic) damping ratio which is often invoked in both geotechnical engineering and seismology. From an experimental point of view a direct validation of the theoretical solutions was illustrated through a comparison with recently published data on the frequency-dependence laws of soil dynamic properties measured using non-resonance column tests.

Since the exact solutions of Kramers-Kronig equations establish a direct functional relationship between material damping ratio and the speed of propagation of seismic waves, they can be profitably used to optimize the efforts in the experimental measurement of small-strain dynamic properties of geomaterials and of their frequency dependence. In this regard the possibility of determining material damping ratio spectrum $D(\omega)$ entirely from phase velocity measurements of P and S waves without making any *a-priori* assumption about a specific form for the function $D(\omega)$, seems particularly attractive.

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Sulle implicazioni del principio di causalità fisica nella propagazione di onde sismiche nei geomateriali

Sommario

Obiettivo di questo articolo è l'illustrazione di alcune importanti implicazioni del principio di causalità fisica nella propagazione di onde sismiche nei geomateriali a bassi livelli di deformazione. In queste condizioni il modello costitutivo più semplice in grado di simulare la capacità esibita dai geomateriali soggetti a eccitazioni dinamiche di piccola ampiezza, di accumulare e dissipare energia di deformazione è quello associato alla viscoelasticità lineare. Un risultato importante previsto da tale teoria sul comportamento dei materiali è che la velocità di fase delle onde P ed S e il rapporto di smorzamento del mezzo non sono quantità indipendenti ma sono legate dalle equazioni di dispersione di Kramers-Kronig, le quali non sono altro che un'asserzione delle condizioni necessarie e sufficienti richieste ad un continuo viscoelastico affinché un impulso che si propaga attraverso di esso soddisfi il principio della causalità fisica. Nell'articolo vengono illustrate sia soluzioni approssimate che rigorose recentemente ottenute delle equazioni di Kramers-Kronig, le quali da un punto di vista matematico costituiscono una coppia di equazioni integrali singolari con nucleo di Cauchy. Le soluzioni rigorose sono state derivate senza alcuna assunzione semplificativa oltre a quella associata alla cosiddetta ipotesi della memoria evanescente, una congettura piuttosto debole e peraltro soddisfatta dai geomateriali la quale assume che il tensore degli sforzi corrente dipende in misura più marcata dalla storia recente delle deformazioni anziché da quella più distante nel tempo. Queste soluzioni rigorose delle equazioni di Kramers-Kronig sono attraenti poiché consentono di calcolare, almeno in principio, il rapporto di smorzamento e la sua dipendenza dalla frequenza a partire dalla dispersione della velocità di fase e inversamente, la dipendenza dalla frequenza della velocità di fase delle onde P and S dallo spettro del rapporto di smorzamento. Pertanto queste soluzioni sono ispiratrici di un nuovo approccio alla determinazione delle proprietà dinamiche dei geomateriali a piccole deformazioni dove solo una funzione del materiale viene misurata. I risultati teorici sono stati validati attraverso dati sperimentali ottenuti da prove condotte in colonna non-risonante eseguiti su terreni a grana-fine. Inoltre si è mostrato come essi siano compatibili con le soluzioni approssimate derivate precedentemente delle equazioni di Kramers-Kronig ed in modo particolare con la ben nota relazione di dispersione ampiamente utilizzata in sismologia e basata sull'ipotesi che il rapporto di smorzamento sia indipendente dalla frequenza nel campo della banda sismica.