A strain softening creep model for the time-dependent behaviour of rocks

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Summary

A visco-plastic rheological model is first illustrated which accounts for the so called primary (reversible) and secondary (plastic) components of the creep deformation. The model is subsequently extended to introduce the effects of tertiary creep. This is done by considering the reduction of shear strength, and apparent viscosity, of the rock with increasing shear deformation, after the onset condition has been reached. Two alternative conditions for the initiation of tertiary creep are discussed. They are directly derived from those adopted in time-independent, strain softening analyses for detecting the onset of the so called shear bands, and the consequent loss of the mechanical resistance of the rock element.

1. Introduction

The possible time-dependent behaviour, or creep, is one of the important mechanical characteristics of the rock mass that should be taken into account in the design of relevant rock engineering works, such as deep tunnels or underground chambers [SAKURAI, 1978].

With reference to a deviatoric stress state constant with time (Fig. 1), this behaviour is customarily subdivided into three main parts [CRISTESCU, 1988]. The so called primary creep, having a substantially reversible nature, which is present at stress levels markedly lower than the ultimate one and is characterized by a strain rate decreasing with time. The secondary creep behaviour shows up at higher stress levels and induces irreversible, plastic deformation in the rock mass. In this case the strain rate tends to stabilize to an almost constant value. Finally, for stress levels approaching the "instantaneous" yielding condition, an increase of the strain rate is observed during time which leads to failure of the rock mass.

This last effect is particularly relevant for evaluating the stability of tunnels and underground openings. In fact, the excavation process induces an increase of shear stresses in the rock mass around the opening, which in turn may cause the development of non reversible strains. This could lead to the onset of tertiary creep, which governs the so called short term deformation of the rock surrounding the tunnel, the "stand up" time of the unsupported portions of the opening and the value of the rock pressure acting on its temporary support. This behaviour, customarily referred to as "squeezing" [BARLA, 1995], implies a volume increase of the rock mass only when the rock itself exhibits dilatancy or when it is associated with a swelling phenomenon. Squeezing is in general a long term effect and is strongly dependent on the in-situ conditions, as well as on the construction process. It can be inhibited by the application of a proper tunnel liner, although this leads to a gradual increase of the rock pressure on the liner after the construction is completed.

In the following, the main features of an isotropic, visco-plastic rheological model allowing for primary and secondary creep are first illustrated. Then, two alternative conditions are discussed for detecting the initiation of tertiary creep, which is seen as a loss of the overall shear resistance of the rock due to its strain softening behaviour. An expression of the tangent visco-plastic constitutive ma-

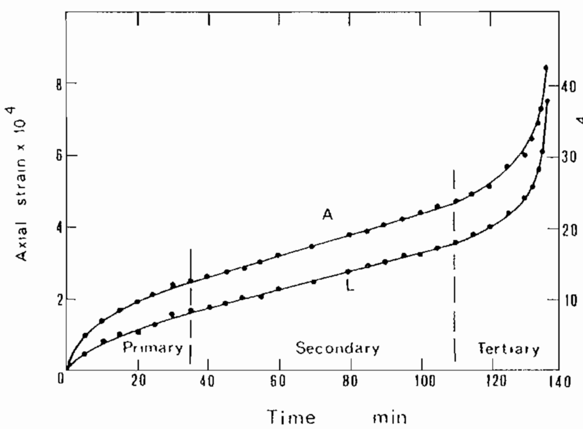

Fig. 1 – Axial (A) and lateral (L) strain vs. time curves from constant load compression tests (after GIODA and CIVIDINI, 1994).

Fig. 1 – Variazione nel tempo delle deformazioni assiale (A) e laterale (L) in prove di compressione a carico costante (da GIODA e CIVIDINI, 1994).

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trix is then obtained for the adopted rheological model. Some comments are also presented on the physical nature of the conditions for tertiary creep and on the necessary developments of this study.

2. Rheological model

The adopted visco-plastic rheological model, see e.g. [GIODA and CIVIDINI, 1996], consists of a visco-elastic Kelvin model, accounting for primary creep, connected in series with a visco-plastic Bingham model accounting for secondary creep (Fig. 2).

The constitutive equation of the visco-elastic component can be expressed in the following form

$$\underline{\sigma}_t = \underline{D}^{ve} \cdot \underline{\varepsilon}_t^{ve} + \underline{V}^{ve} \cdot \dot{\underline{\varepsilon}}_t^{ve} \quad , \qquad (1)$$

where $\underline{D}^{ve}$ and $\underline{V}^{ve}$ are the elastic and viscous constitutive matrices, $\underline{\varepsilon}^{ve}$ is the recoverable creep strain and t denotes the time. A superposed dot means time derivative. Note that, in the isotropic case, matrix $\underline{V}^{ve}$ has the same structure of matrix $\underline{D}^{ve}$ where the elastic parameters, e.g. shear and bulk moduli, are replaced by the corresponding shear and bulk viscosity coefficients.

As to the visco-plastic behaviour, let consider first for sake of simplicity the case in which the material parameters of the Bingham model are constant and its frictional component is perfectly plastic.

A basic hypothesis adopted in the following is that the Newton element (dashpot) carries only deviatoric stresses, while De St.Venant (frictional) element can support both volumetric and deviatoric stresses. On the basis of this assumption a stress state exceeding the resistance of the frictional element can be subdivided into its octahedral component σ_{oct}, that acts on the frictional element, and its deviatoric part $\underline{s}$ which, in turn, is expressed as the sum of the components carried by the dashpot $\underline{s}^{cr}$ and by De St.Venant element $\underline{s}^{y}$

$$\underline{\sigma}_t = \underline{m}(\sigma_{oct})_t + \underline{s}_t^{cr} + \underline{s}_t^{y} \quad . \qquad (2)$$

In the above equation, the entries of vector $\underline{m}$ are equal to 1, if they correspond to normal stresses,

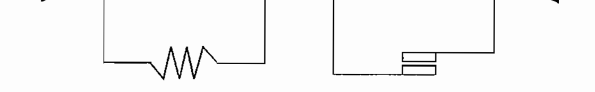

Fig. 2 – Visco-plastic rheological model accounting for primary and secondary creep.

Fig. 2 – Modello reologico visco-plastico che considera viscosità di tipo primario e secondario.

or to 0, if they correspond to shear stresses. Note that $\underline{s}^{y}$ is readily evaluated on the basis of the octahedral stress since the resulting stress $\underline{\sigma}^{y}$

$$\underline{\sigma}_t^{y} = \underline{m}(\sigma_{oct})_t + \underline{s}_t^{y} \qquad (3)$$

must fulfill the yield condition F adopted for De St.Venant element.

The visco-plastic (creep) strain rate $\dot{\underline{\varepsilon}}^{cr}$ is subdivided into its volumetric $\dot{\varepsilon}_{vol}^{cr}$ and deviatoric $\dot{\underline{\sigma}}^{cr}$ parts, which are represented by orthogonal vectors

$$\dot{\underline{\varepsilon}}_t^{cr} = \underline{m} \left(\frac{\dot{\varepsilon}_{vol}^{cr}}{3} \right)_t + \dot{\underline{e}}_t^{cr} \quad , \qquad (4)$$

$$\underline{m}^T \dot{\underline{\varepsilon}}^{cr} = 0 \quad . \qquad (5)$$

The constitutive relationship for the dashpot can be expressed in the following form

$$\underline{s}_t^{cr} = \underline{V}^{cr} \cdot \dot{\underline{e}}_t^{cr} \quad . \qquad (6)$$

Note that due to the assumption of a purely deviatoric behaviour, the viscosity matrix $\underline{V}^{cr}$ depends on only one deviatoric viscosity coefficient.

Since the viscous and frictional elements are subjected to the same deformation, plastic and creep strains coincide. Consequently the plastic flow rule can be expressed in terms of the creep strain rate $\dot{\underline{\varepsilon}}^{cr}$ and of the rate of the plastic multiplier $\dot{\underline{\lambda}}$ through the relationship

$$\dot{\underline{\varepsilon}}^{cr} = \dot{\lambda}_t \cdot \underline{q}_t \quad , \qquad (7)$$

where $\underline{q}_t$ is the gradient of the plastic potential Q determined at $\underline{\sigma}_t^{y}$

$$\underline{q}_t = \frac{\partial Q}{\partial \underline{\sigma}_t^{y}} \quad . \qquad (8)$$

The finite element formulation of the creep problem and an iterative scheme for its time integration have been discussed elsewhere [GIODA and CIVIDINI, 1996] and will not be recalled here for sake of briefness.

3. Conditions for the initiation of tertiary creep

Two hypotheses on the onset of softening [VARDOULAKIS and SULEM, 1995] are here considered, which lead to different numerical approaches [STERPI, 1997] for the analysis of tertiary creep effects.

According to a first approach this phenomenon initiates when a given limit is reached by a proper scalar measure of the irreversible strains, represented e.g. by the second invariant of their deviatoric part or by the plastic strain work. When this condition is fulfilled in a rock element, the values of its shear strength parameters gradually decrease with increasing plastic strains, until reaching their minima (ultimate, or residual, values). During this process, the local elastic moduli of the material, and its apparent viscosity, can be reduced as well.

In the second case, strain softening is caused by a loss of uniqueness of the solution of the governing equations which in turn leads to the formation of a discontinuity plane, or shear band, within the rock element. Note that this condition expresses the initiation of softening in terms of the stress components, while the first approach relates the same condition to the accumulated plastic strains. To account for the change of the material structure within the shear band, the values of the frictional and cohesive shear strength parameters are gradually reduced with increasing deviatoric plastic strains, following the before mentioned procedure. This second approach for detecting the onset of softening follows the procedure for elasto-plastic, rate independent problems that was proposed by ORTIZ *et al.* [1987] and was adopted in [STERPI *et al.*, 1995].

Consider a discontinuity plane, in a reference system $\underline{x}$, having an unknown inclination defined by the unit vector $\underline{n}$ normal to it. The rates are denoted by a superposed dot, while <.> indicates the difference between the values of the same variable evaluated on the two sides of the discontinuity.

The equilibrium conditions for the stress rate $\dot{\underline{\sigma}}$ across the plane read

$$\underline{N}T \cdot <\dot{\underline{\sigma}}> = \underline{0} \cdot, \qquad (9)$$

where the entries of $\underline{N}$ are the direction cosines of vector $\underline{n}$.

The difference $<\dot{\underline{u}}>$ between the displacement rates $\dot{\underline{u}}$ on the two sides of the discontinuity can be expressed as

$$<\dot{\underline{u}}> = \dot{\underline{u}}^{+} - \dot{\underline{u}}^{-} = \dot{g}\underline{m} \cdot (\underline{n}^{T} \cdot \underline{x}). \qquad (10)$$

In Eq. 10, $(\underline{n}^{T} \cdot \underline{x})$ represents the distance from the plane; $\dot{g}$ is the amplitude rate and $\underline{m}$ is the unknown unit vector that defines the direction of vector $<\dot{\underline{u}}>$. Eq. (10) shows that the displacement field remains continuous after the onset of localization and that the difference between the displacement increments on the two sides of the plane varies linearly with the distance from it.

The gradient of $<\dot{\underline{u}}>$ can be expressed by the Maxwell's form of the compatibility conditions [THOMAS, 1961], leading to the following matrix expression of the "jumps" of the strain rates across the discontinuity

$$<\dot{\underline{\varepsilon}}> = \dot{g}\underline{N} \cdot \underline{m}. \qquad (11)$$

In addition, a general relationship for the constitutive law is considered

$$\dot{\underline{\sigma}} = \underline{D}^{vp} \cdot \dot{\underline{\varepsilon}}, \qquad (12)$$

where the tangent visco-plastic matrix $\underline{D}^{vp}$ will be evaluated in the following.

Assuming that the material on the two sides of the discontinuity follows the same incremental behaviour [RICE and RUDNICKI, 1980], the same relationship can be written for the jumps of the stress and strain rates

$$<\dot{\underline{\sigma}}> = \underline{D}^{vp} \cdot <\dot{\underline{\varepsilon}}^{e}> \ . \qquad (13)$$

By substituting Eq. (11) into Eq. (13) and taking into account Eq. (9), the following homogeneous expression is arrived at

$$[\underline{N}^{T} \cdot \underline{D}^{vp} \cdot \underline{N}] \cdot \underline{m}\dot{g} = 0, \qquad (14)$$

which shows that localization takes place only if the determinant of the matrix within square brackets vanishes.

As far as the material remains in an elastic state, this condition has no real solutions. When a yielding state is reached, in the presence of a non associated plastic flow rule the condition can lead to real solutions also in the case of perfectly plastic or positive hardening behaviour [RUDNICKI and RICE, 1975]. Observing that in the former case the localization condition depends only on the stress state and on the material parameters, it is possible to evaluate the stress states in which the above condition is fulfilled [STERPI, 1999].

As previously mentioned, once the criterion for the initiation of tertiary creep is fulfilled, a loss of strength (and apparent viscosity) takes place with increasing permanent deformation. In particular, it is assumed that the shear strength parameters are linear functions of a measure of the irreversible deformation represented by the square root of the second invariant of the deviatoric plastic strains. The parameters remain unchanged, and equal to their peak values, until the previously discussed condition is fulfilled. Then, they are reduced with increasing plastic deformation until their residual values are reached.

4. Visco-plastic constitutive matrix

A tangent visco-plastic matrix for the constitutive law of Eq. 12, is derived in the following with reference to a simplified rheological model (Fig. 3), which accounts only for secondary and tertiary creep. In fact, primary (visco-elastic) creep can be neglected without introducing a limit to the development of plastic shear strain discontinuities.

The instantaneous behaviour of the rheological model in Fig. 3 can be written as

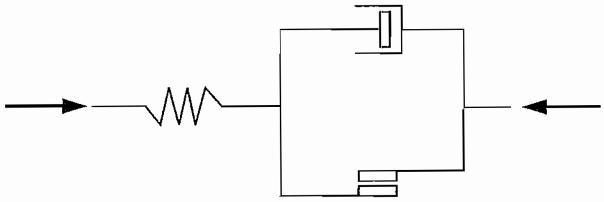

Fig. 3 – Visco-plastic rheological model accounting for secondary creep.
Fig. 3 – Modello reologico visco-plastico che considera viscosità di tipo secondario.

$$\dot{\underline{\sigma}} = \underline{D} \cdot \dot{\underline{\varepsilon}}^e , \qquad (15)$$

where the 'elastic' deformation rate $\dot{\underline{\varepsilon}}^e$ depends on the total $\dot{\underline{\varepsilon}}$ and creep $\dot{\underline{\varepsilon}}^{cr}$ deformation rates through the condition

$$\dot{\underline{\varepsilon}} = \dot{\underline{\varepsilon}}^e \cdot \dot{\underline{\varepsilon}}^{cr} . \qquad (16)$$

By substituting Eq. (15) in Eq. (16) the following relation is arrived at

$$\dot{\underline{\varepsilon}} - \underline{D}^{-1} \dot{\underline{\sigma}} - \dot{\underline{\varepsilon}}^{cr} = \underline{0} \qquad (17)$$

By considering a time increment Δt, starting from time t, and under the hypothesis that within it stress and strain rates are constant, the incremental Eq. 17 can be written in term of finite stress and strain increments

$$\Delta\underline{\varepsilon}_t - \underline{D}^{-1} \Delta\underline{\sigma}_t - \Delta\underline{\varepsilon}_t^{cr} = \underline{0} \qquad (18)$$

Following the same approach, the dashpot constitutive law (6) and the flow rule (7) read

$$\Delta\underline{e}_t^{cr} - = \Delta t[\underline{V}^{cr}]^{-1} \cdot [(1-\theta)\underline{s}_t^{cr} + \theta\underline{s}_{t+\Delta t}^{cr}] , \qquad (19)$$

$$\Delta\underline{\varepsilon}_t^{cr} - = \Delta\lambda_t [(1-\theta)\underline{q}_t + \theta\underline{q}_{t+\Delta t}] , \qquad (20)$$

where coefficient θ ranges between 0 and 1 and allows for both an explicit integration scheme, when the variable depends only on quantities at the end of the previous time increment (θ=0), or an implicit integration scheme (θ>0).

Note that in Eq. 19, the deviatoric stress $\underline{s}^{cr}$ acting on the dashpot can be readily evaluated from Eqs. 2 and 3, so that the deviatoric creep strain increment $\Delta\underline{e}^{cr}$ is expressed as function of the difference between the total stress $\underline{\sigma}$ and the yield stress $\underline{\sigma}^y$.

Volumetric and deviatoric creep strain rates (Eq. 4) are obtained, on the basis of the flow rule (7), as

$$(\dot{\varepsilon}_{vol}^{cr})_t = \dot{\lambda}_t\,(tr\underline{q}_t) , \qquad (21)$$

$$\dot{\underline{e}}_t^{cr} = \dot{\lambda}_t \left(\underline{q}_t - \frac{m}{3}(tr\underline{q}_t)\right) , \qquad (22)$$

where (tr $\underline{q}$) is the summation of the derivatives of plastic potential Q with respect to the normal components of stress. In particular Eq. 22, in terms of finite increments, reads

$$\Delta\underline{e}_t^{cr} = \Delta\lambda_\tau \left\{(1-\theta)\left[\underline{q}_t - \frac{m}{3}(tr\,\underline{q}_t)\right] + \theta\left[\underline{q}_{t+\Delta t} - \frac{m}{3}(tr\,\underline{q}_{t+\Delta t})\right]\right\} \qquad (23)$$

Two governing equations are then arrived at, respectively by substituting expression (20) in (18) and by equating expression (19) to (23).

By differentiating the two equations within the time increment Δt [ZIENKIEWICZ and TAYLOR, 1987], a governing system is derived in which the unknown variables are the infinitesimal variations $\delta\underline{\varepsilon}$, $\delta\underline{\sigma}$, $\delta\underline{\sigma}^y$ and δλ. Among these, variation $\delta\underline{\sigma}^y$ is eliminated by means of the consistency condition

$$\left(\frac{\partial F}{\partial \underline{\sigma}_t^y}\right)^T \cdot \delta\underline{\sigma}_t^y = 0 \qquad (24)$$

and the system is reduced to the following form

$$\begin{bmatrix} \underline{A}_{11} & \underline{a}_{12} \\ \underline{a}_{21}^T & a_{22} \end{bmatrix}_t \cdot \begin{bmatrix} \delta\underline{\sigma} \\ \delta\lambda \end{bmatrix}_t = \begin{bmatrix} \delta\underline{\varepsilon} \\ 0 \end{bmatrix}_t , \qquad (25)$$

where a_{22} and the entries of $\underline{A}_{11}$, $\underline{a}_{12}$, and $\underline{a}_{21}$ are functions of the time increment Δt, of the mechanical parameters, through $\underline{D}$ and $\underline{V}^{cr}$, and of the first and second derivatives of yield function F and plastic potential Q with respect to the stress components.

Finally, by elimination of δλ between the two governing equations (25), a relation is obtained between total stress and total strain infinitesimal variations

$$\underline{C}_t^{vp} \cdot \delta\underline{\sigma}_t = \delta\underline{\varepsilon}_t , \qquad (26)$$

from which, in case of non-singularity of matrix $\underline{C}^{vp}$, an incremental constitutive relationship is arrived at in the form of Eq. 12.

5. Conclusions

A rheological model has been illustrated which accounts for the so called primary, secondary and tertiary creep stages. In particular, tertiary creep is viewed as an effect of the strain softening behaviour of the rock and is introduced by reducing the values of the shear strength parameters, and of the apparent viscosity, with increasing plastic deformation, after a proper initiation condition has been reached.

Two conditions of this kind have been discussed which lead to criteria for detecting the onset of softening based, respectively, on the accumulated plastic strains and on the current stress level.

The two criteria are based on hypotheses having different physical nature. In addition, the second criterion requires a computational effort markedly higher than that required by the first one. These differences suggest some further investigation concerning, from the one hand, the application of these approaches to practical problem and, from the other hand, the comparison between the two solutions and the available experimental information.

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Un modello a comportamento "strain softening" per l'analisi della risposta deformativa di rocce, dipendente dal tempo

Sommario

Il modello reologico visco-plastico illustrato considera, di base, le due componenti della deformazione di "creep", dette primaria (reversibile) e secondaria (plastica). Il modello viene qui esteso allo scopo di considerare anche gli effetti del "creep" di tipo terziario. All'aumentare delle deformazioni deviatoriche e dopo aver raggiunto una condizione di inizializzazione, si introduce una riduzione di resistenza al taglio e di viscosità apparente della roccia. Vengono discusse due condizioni alternative per l'innesco del "creep" terziario, direttamente derivate dalle condizioni adottate per l'analisi della formazione di bande di taglio, e della conseguente perdita di resistenza meccanica nell'elemento di roccia, per materiali a comportamento "strain softening" indipendente dal tempo.